

# Linear dimension-free estimates for the Hermite-Riesz transforms <sup>\*</sup>

Oliver Dragičević and Alexander Volberg

November 5, 2008

## Abstract

We utilize the Bellman function technique to prove a bilinear dimension-free inequality for the Hermite operator. The Bellman technique is applied here to a non-local operator, which at first did not seem to be feasible. As a consequence of our bilinear inequality one proves dimension-free boundedness for the Riesz-Hermite transforms on  $L^p$  with linear growth in terms of  $p$ . A feature of the proof is a theorem establishing  $L^p(\mathbb{R}^n)$  estimates for a class of spectral multipliers with bounds independent of  $n$  and  $p$ . Connections with known results on the Heisenberg group as well as with results for Hilbert transform along the parabola are also explored. We believe our approach is quite universal in the sense that one could apply it to a whole range of Riesz transforms arising from various differential operators. As a first step towards this goal we prove our dimension-free bilinear embedding theorem for quite a general family of Schrödinger semigroups.

## 1 Introduction

The purpose of this paper is to prove bilinear  $L^p \times L^q \rightarrow L^1$  embeddings associated with the Hermite operator on  $\mathbb{R}^n$ . Similar results were proved in [5] for the usual Laplacian and the Ornstein-Uhlenbeck operator. In this paper we manage to apply the same method (Bellman functions) and treat operators with potentials, such as the Hermite Laplacian, which is a novelty compared to [5]. All of our embedding theorems are dimension-free

---

<sup>\*</sup>The first author was supported by the Ministry of Higher Education, Science and Technology of Slovenia (research program Analysis and Geometry, contract no. P1-0291). The second author was supported by NSF grant DMS 0501067.

and exhibit linear estimates in terms of  $p$ . This feature is due to special properties of the concrete Bellman function we use. It is further exploited in [6] where the embedding theorem is proved for extensions generated by second-order Schrödinger-type operators in divergence form with real coefficients (i.e. regarding Kato's problem with real matrix).

As the main yet simple consequence of our embedding theorem we give dimension-free  $L^p$  estimates of Riesz transforms for the Hermite semigroup. Our approach already gave such estimates for euclidean and Ornstein-Uhlenbeck semigroups, see [5], and there we announced analogous results for a wider class of differential operators, provided only that their spectral properties are not “too singular”, so that we can construct an efficient dimension-free passage from the embedding theorems. In the Hermite case the latter appear here, to the best of our knowledge, for the first time in literature.

Regarding the applications to the Riesz transforms we should say that for euclidean, Ornstein-Uhlenbeck and Hermite semigroups our approach offers a unified way of treating various vector-valued operators of this kind and obtaining dimension-free estimates. Apart from this uniformity, it yields explicit estimates of the  $L^p$  norms for Riesz transforms in terms of  $p$ . In the case considered here – the one of the Hermite operator – we establish linear behaviour with respect to  $p$ . This result seems to be new and we believe it to be sharp. We also prove dimension-free estimates of iterations of Riesz transforms. In order to yield such estimates we present bounds on  $L^p(\mathbb{R}^n)$  for spectral multipliers arising from functions analytic at infinity. These bounds are absolute, i.e. independent of  $n$  and  $p$ . Again, we believe that exactly the same treatment can be applied to a wide class of differential operators (infinitesimal generators), under the condition that their spectral properties are not “too singular”.

We tried to make the paper as self-contained as possible, given that we were naturally obliged to refer to some basic facts.

## Statement of main results

Take  $p \in (1, \infty)$  and denote by  $q$  its conjugate exponent  $p/(p-1)$ . We will use the notation  $p^* := \max\{p, q\}$ .

The *Hermite operator*  $L$  is, for a test function  $u$  on  $\mathbb{R}^n$ , defined as

$$Lu(x) = -\Delta u(x) + |x|^2 u(x).$$

An equivalent description of  $L$  is given by

$$L = \frac{1}{2} \sum_{j=1}^n (\mathcal{A}_j \mathcal{A}_j^* + \mathcal{A}_j^* \mathcal{A}_j), \quad (1)$$

where

$$\mathcal{A}_j = -\frac{\partial}{\partial x_j} + x_j \quad \text{and} \quad \mathcal{A}_j^* = \frac{\partial}{\partial x_j} + x_j \quad (2)$$

are the *creation* and *annihilation* operator, respectively. The operator  $L$  is positive, meaning that  $\langle Lu, u \rangle \geq 0$ . For a thorough discussion of  $L$  we refer the reader to [30].

We will be dealing with the operator semigroup  $\{P_t := e^{-t\sqrt{L}}\}_{t>0}$ . Denote  $\tilde{u}(x, t) = P_t u(x)$ . This function solves on  $\mathbb{R}^n \times (0, \infty)$  the differential equation

$$\left(\frac{\partial^2}{\partial t^2} - L\right)\tilde{u} = 0, \quad (3)$$

with  $\tilde{u}(x, 0) = u(x)$ .

For a given smooth  $\mathbb{C}^N$ -valued function  $\phi = (\phi_1(x, t), \dots, \phi_N(x, t))$  on  $\mathbb{R}^n \times (0, \infty)$  denote

$$\begin{aligned} \|\phi\|_*^2 &= \left|\frac{\partial \phi}{\partial t}\right|^2 + \sum_{j=1}^n \left|\frac{\partial \phi}{\partial x_j}\right|^2 + |x|^2 |\phi(x, t)|^2 \\ &= \left|\frac{\partial \phi}{\partial t}\right|^2 + \frac{1}{2} \sum_{j=1}^n (|\mathcal{A}_j \phi(x, t)|^2 + |\mathcal{A}_j^* \phi(x, t)|^2). \end{aligned} \quad (4)$$

Here, as usual,

$$\frac{\partial \phi}{\partial x_j} = \left(\frac{\partial \phi_1}{\partial x_j}, \dots, \frac{\partial \phi_N}{\partial x_j}\right).$$

The  $L^p$  norm of a  $\mathbb{C}^N$ -valued test function  $\psi$  on  $\mathbb{R}^n$  is of course  $(\int_{\mathbb{R}^n} \|\psi(x)\|_{\mathbb{C}^N}^p dx)^{1/p}$ .

We are ready to state the key proposition, which can be considered as the bilinear embedding theorem (or bilinear Littlewood-Paley theorem) for the Hermite extension.

**Theorem 1.** *There is an absolute constant  $C > 0$  such that for arbitrary natural numbers  $M, N, n$ , any pair  $f : \mathbb{R}^n \rightarrow \mathbb{C}^M$  and  $g : \mathbb{R}^n \rightarrow \mathbb{C}^N$  of  $C_c^\infty$  test functions and any  $p > 1$  we have*

$$\int_0^\infty \int_{\mathbb{R}^n} \|P_t f(x)\|_* \|P_t g(x)\|_* dx t dt \leq C(p^* - 1) \|f\|_p \|g\|_q.$$

Note that these estimates are dimension free.

### Schrödinger operators with positive potentials

The preceding theorem can without changing the statement be generalized to a large class of Schrödinger operators  $L = -\Delta + V(x)$ . Here  $V$  is a non-negative function on  $\mathbb{R}^n$ . By  $P_t$  denote the (Poisson) operator semigroup whose infinitesimal generator is  $L^{1/2}$ . Let  $\mathcal{K}_t$  be the heat kernel, i.e. the kernel associated to the semigroup generated by  $L$ . Assume the following conditions on  $V$ :

(a) Kato's inequality

$$\mathcal{K}_t(x, y) \leq Ct^{-\frac{n}{2}} e^{-\frac{a}{t}|x-y|^2}$$

and the kernel being sub-probability, i.e.

$$\int_{\mathbb{R}^n} \mathcal{K}_t(x, y) dy \leq 1.$$

for all  $x \in \mathbb{R}^n$ . Also,  $\mathcal{K}_t$  needs to be non-negative.

(b) Gradient estimates for the heat kernel:

$$\begin{aligned} \left| \frac{\partial}{\partial x_j} \mathcal{K}_t(x, y) \right| &\leq Ct^{-\frac{n+1}{2}} e^{-\frac{a}{t}|x-y|^2} \\ \left| \frac{\partial}{\partial t} \mathcal{K}_t(x, y) \right| &\leq Ct^{-\frac{n}{2}-1} e^{-\frac{a}{t}|x-y|^2}. \end{aligned}$$

(c) If  $g \in C_c^\infty$  then

$$\begin{aligned} \lim_{t \rightarrow 0} \int_{\mathbb{R}^n} \left| t \frac{\partial P_t g}{\partial t}(x) \right| dx &= 0 \\ \lim_{t \rightarrow \infty} \int_{\mathbb{R}^n} t \frac{\partial P_t g}{\partial t}(x) dx &= 0. \end{aligned}$$

(d) For any bounded, non-negative, compactly supported function  $\varphi$  and some  $C_0 > 0$  which does not depend on  $n$ ,

$$\int_{\mathbb{R}^n} \int_0^\infty P_t \varphi(x) t dt V(x) dx \leq C_0 \|\varphi\|_1.$$

The conditions which imply (a) were studied in [22], for example. In general there exists a vast literature on estimates of heat kernels.

Under the conditions (a) – (d) we get exactly the same statement as in Theorem 1. We emphasize that the constants  $C, a$  from the conditions (a) and (b) are allowed to be arbitrary (meaning also dependent on the dimension  $n$ ), and still they allow us to deduce dimension-free estimates in the analogue of Theorem 1 for general potential  $V$  as above.

## Riesz transforms

We mentioned above that as an application of the embedding theorem we obtain an  $L^p$  estimate of the corresponding *Riesz transforms*, introduced in [30] as

$$R_j = \mathcal{A}_j L^{-\frac{1}{2}}, \quad R_j^* = \mathcal{A}_j^* L^{-\frac{1}{2}}, \quad j = 1, \dots, n.$$

A proof establishing their  $L^p$  boundedness can be found in [30]. See also [29]. This result has later been enhanced, as we describe now.

Define

$$\mathbf{R}f = \left( \sum_{j=1}^n |R_j f|^2 + \sum_{j=1}^n |R_j^* f|^2 \right)^{\frac{1}{2}}.$$

It was proved in [11] and shortly afterwards in [15] that the  $L^p$  bounds for  $\mathbf{R}$  admit estimates from above with constants not depending on the dimension. Apparently, however, the authors of [11] and [15] missed the fact that this result can be deduced, by means of transference, from an earlier work [4] addressing analogous questions on the Heisenberg group. Moreover, the approach from [4] also permits explicit information about behaviour of the estimates with respect to  $p$ ; it seems [11] and [15] do not contain that. Namely, the main result of [4] draws a close connection between Heisenberg-Riesz transforms and the Hilbert transform along the parabola in  $\mathbb{R}^2$ . The  $L^p$  boundedness of the latter has been known for a long time [24], yet only not long ago, owing to several highly nontrivial results due to Seeger, Tao and Wright [21, 25, 26], has there been major improvement as to the behaviour of estimates in terms of  $p$ . As a result it is possible to obtain, for arbitrary  $\varepsilon > 0$ , the estimate  $\|\mathbf{R}f\|_p \leq C_\varepsilon p^{1+\varepsilon} \|f\|_p$ . We shall devote section 4.3 to the purpose of explaining these connections and developments.

As a consequence of our Theorem 1 we are able to further sharpen this result and obtain linear estimates, which we believe to be optimal.

**Corollary 1.** *There exists  $C > 0$  such that for any  $1 < p < \infty$ ,  $n \in \mathbb{N}$  and  $f \in L^p(\mathbb{R}^n)$ ,*

$$\|\mathbf{R}f\|_p \leq C(p^* - 1) \|f\|_p. \quad (5)$$

This linear inequality either gives indirect evidence in favour of the Hilbert transform along parabola admitting linear  $L^p$  estimates (see (75) in section 4.3), or else indicates that the method involving the Bellman function is yet sharper than the passage through the Heisenberg groups described above. We believe the second option to be more likely.

The estimate (5) can be generalized to encompass iterations of Riesz transforms. A theorem of the same type was proved recently in [15], but apparently without the numerical estimate of  $C(p)$ . Let us formulate our result rigorously. Write

$$\mathcal{C}_d = \{\text{compositions of } d \text{ operators among } R_1, \dots, R_n, R_1^*, \dots, R_n^*\}.$$

**Corollary 2.** *Assuming conditions and notation as in Corollary 1,*

$$\left\| \left( \sum_{R^d \in \mathcal{C}_d} |R^d f|^2 \right)^{1/2} \right\|_p \leq C^d (p^* - 1)^d \|f\|_p.$$

## Spectral multipliers

It is worth mentioning that in order to prove the above corollaries we come up with another result which we believe to be of independent interest. It deals with  $L^p$ -boundedness of spectral multipliers for Hermite expansions. Theorems of this type were obtained in [16] and [30]. Our theorem is different in terms of the assumptions laid on the multiplier and also because we obtain absolute bounds for the norms, i.e. such which do not depend on  $n$  or  $p$ . Further discussion of the theorem and its proof are to be found in section 4.1.

**Theorem 2.** *Let function  $\Psi$  be analytic at infinity. Then for all  $p \in [1, \infty]$  and all  $n \in \mathbb{N}$ , the operator  $\Psi(L)$  is bounded on  $L^p(\mathbb{R}^n)$  by a constant which depends only on  $\Psi$ , i.e. it does not depend neither on  $n$  nor  $p$ .*

**Acknowledgements.** We are grateful to Giancarlo Mauceri and Stefano Meda for discussions on spectral multipliers for the Hermite operator. Our deep gratitude goes to Fulvio Ricci, Adam Sikora, José Luis Torrea and Françoise Lust-Piquard for many helpful explanations regarding the Hermite semigroup. We also thank Jim Wright for conveying to us recent results about Hilbert transforms along curves.

Finally, we wish to thank Centro Di Ricerca Matematica Ennio De Giorgi, Pisa, for the hospitality extended to us for the two weeks in March 2007.

## 2 Bellman function

Throughout the section we work with  $p \geq 2$ ,  $q = p/(p-1)$  and  $\delta = q(q-1)/8$ . Observe that  $\delta \sim (p-1)^{-1}$ .

The crucial part in our proofs will be played by the function  $Q$ , given by

$$Q(\zeta, \eta, Z, H) = 2(Z + H) - |\zeta|^p - |\eta|^q - \delta \tilde{Q}(\zeta, \eta), \quad (6)$$

where

$$\tilde{Q}(\zeta, \eta) = \begin{cases} |\zeta|^2 |\eta|^{2-q} & ; \quad |\zeta|^p \leq |\eta|^q \\ \frac{2}{p} |\zeta|^p + \left(\frac{2}{q} - 1\right) |\eta|^q & ; \quad |\zeta|^p \geq |\eta|^q \end{cases}.$$

Note that the definition of  $Q$  depends on  $p$ . Such a function, defined on a subdomain in  $\mathbb{R}^4$ , was first introduced by F. Nazarov and S. Treil [19]. Here it is defined in the domain

$$\Omega := \{(\zeta, \eta, Z, H) \in \mathbb{C}^M \times \mathbb{C}^N \times \mathbb{R} \times \mathbb{R}; |\zeta|^p \leq Z, |\eta|^q \leq H\}.$$

Function  $Q$  is in  $C^1(\Omega)$  and its second derivatives are continuous except on  $\gamma := \{|\zeta|^p = |\eta|^q\}$  but everywhere in  $\Omega$  (including  $\gamma$ ) these second derivatives or their one-sided limits can be estimated in a way which suits our purposes very well.

The Hessian matrix of  $Q$  is denoted by  $d^2Q$ . Thus  $d^2Q$  is a matrix-valued function which maps vector  $\omega \in \Omega$  into the matrix with entries  $\frac{\partial^2 Q}{\partial \alpha \partial \beta}(\omega)$ , where  $\alpha$  and  $\beta$  range over  $\zeta_j, \overline{\zeta_j}, \eta_k, \overline{\eta_k}, Z, H$  for  $j = 1, \dots, M$ ,  $k = 1, \dots, N$ .

**Theorem 3.** Choose  $\omega = (\zeta, \eta, Z, H) \in \Omega$ . Then

$$(i) \quad Q(\omega) \leq 2(Z + H).$$

There exists  $\tau = \tau(|\zeta|, |\eta|) > 0$  such that

$$(ii) \quad -d^2Q(\omega) \geq \delta(\tau|d\zeta|^2 + \tau^{-1}|d\eta|^2)$$

$$(iii) \quad Q(\omega) - \omega \cdot \nabla Q(\omega) \geq \delta(\tau|\zeta|^2 + \tau^{-1}|\eta|^2).$$

To clarify the notation let us say that by (ii) we mean that

$$\langle -d^2Q(\omega)w, w \rangle \geq \delta(\tau|w_1|^2 + \tau^{-1}|w_2|^2)$$

for all  $w = (w_1, w_2, w_3, w_4) \in \mathbb{C}^M \times \mathbb{C}^N \times \mathbb{R} \times \mathbb{R}$ .

When constructing their “scalar” Bellman function in [19], Nazarov and Treil aimed at property (i) and a weaker version of (ii), namely  $-d^2Q(\omega) \geq 2\delta|d\zeta||d\eta|$ . They apparently did not study anything like (iii). It does not seem that (i) and (ii) imply (iii). It was thus a considerable surprise for us to see that

(iii) is nevertheless also true, especially since we can prove it with the same  $\tau$  as in (ii), which is essential for our applications (see Lemma 1).

**Proof.** The inequality (i) is obvious. Let us first prove (ii). Consider

$$\Phi(\zeta, \eta) = |\zeta|^p + |\eta|^q + \delta \begin{cases} |\zeta|^2 |\eta|^{2-q} & ; \quad |\zeta|^p \leq |\eta|^q \\ \frac{2}{p} |\zeta|^p + \left(\frac{2}{q} - 1\right) |\eta|^q & ; \quad |\zeta|^p \geq |\eta|^q \end{cases}.$$

Here (unlike before) we think of  $\zeta$  and  $\eta$  as *real* vectors in real  $l_{2M}^2, l_{2N}^2$  correspondingly. Also,  $|\cdot|$  denotes the  $l^2$  norm of the corresponding vector. We want to take a look at  $d^2\Phi$  (the Hessian of  $\Phi$ , its second differential form). To do that write  $\Phi = \phi \circ U$ , where  $U(\zeta, \eta) = (|\zeta|, |\eta|)$  and  $\phi$  is a function of two non-negative real variables given by

$$\phi(u, v) = u^p + v^q + \delta \begin{cases} u^2 v^{2-q} & ; \quad u^p \leq v^q \\ \frac{2}{p} u^p + \left(\frac{2}{q} - 1\right) v^q & ; \quad u^p \geq v^q \end{cases}. \quad (7)$$

Denote by  $e_\zeta$  the unit vector  $\zeta/|\zeta|$  and by  $P_\zeta$  the orthogonal projection onto the orthogonal complement of  $e_\zeta$ :  $P_\zeta h = h - \langle h, e_\zeta \rangle e_\zeta$ . Then an easy direct computation gives (see [19])

$$d|\zeta| = \langle d\zeta, e_\zeta \rangle \quad \text{and} \quad d^2|\zeta| = \frac{|P_\zeta d\zeta|^2}{|\zeta|}.$$

The same for  $\eta$ . To clarify the notation, these formulæ are to be understood in the sense that if  $f(a) = |a|$ , then, for  $a \neq 0$ ,  $df(a)h = \langle h, e_a \rangle$  and  $\langle d^2 f(a)h, h \rangle = |P_a h|^2 / |a|$ .

The application of the chain rule gives

$$\begin{aligned} d^2\Phi_{\zeta, \eta}(d\zeta, d\eta) &= d^2\phi_{|\zeta|, |\eta|}(\langle d\zeta, e_\zeta \rangle, \langle d\eta, e_\eta \rangle) + \frac{\partial \phi}{\partial u}(|\zeta|, |\eta|) d^2|\zeta| + \frac{\partial \phi}{\partial v}(|\zeta|, |\eta|) d^2|\eta| \\ &= d^2\phi_{|\zeta|, |\eta|}(\langle d\zeta, e_\zeta \rangle, \langle d\eta, e_\eta \rangle) + \frac{\partial \phi}{\partial u}(|\zeta|, |\eta|) \frac{|P_\zeta d\zeta|^2}{|\zeta|} + \frac{\partial \phi}{\partial v}(|\zeta|, |\eta|) \frac{|P_\eta d\eta|^2}{|\eta|}. \end{aligned} \quad (8)$$

Denote  $A = |d\zeta|$ . Since  $|\langle dh, e_h \rangle|^2 + |P_h dh|^2 = |dh|^2$  for any vectors  $h, dh$ , it follows that  $\langle d\zeta, e_\zeta \rangle = Aa$  and  $|P_\zeta d\zeta|^2 = A^2(1 - a^2)$  for some  $a \in [-1, 1]$ . In the same way (related to  $\eta, d\eta$ ) introduce  $B, b$ . Write also



$u = |\zeta|$ ,  $v = |\eta|$ . Therefore our task is to find as good lower estimates as possible for the expression

$$F_{u,v}(A, a, B, b) := d^2\phi_{u,v}(Aa, Bb) + \frac{\partial\phi}{\partial u}(u, v) \frac{A^2(1-a^2)}{u} + \frac{\partial\phi}{\partial v}(u, v) \frac{B^2(1-b^2)}{v}.$$

First consider the case  $u^p \geq v^q$ . Then, by (7),

$$\phi = \left(1 + \frac{2}{p}\delta\right)u^p + \left(1 + \left(\frac{2}{q} - 1\right)\delta\right)v^q,$$

therefore

$$F_{u,v}(A, a, B, b) = (p+2\delta)[1+(p-2)a^2]A^2u^{p-2} + (q+(2-q)\delta)[1-(2-q)b^2]B^2v^{q-2}.$$

Note that the assumption  $u^p \geq v^q$  implies  $v^{q-2} \geq u^{2-p}$ . Moreover, since  $1 + (p-2)a^2 \geq 1$  and  $1 - (2-q)b^2 \geq q-1$ ,

$$\begin{aligned} F_{u,v}(A, a, B, b) &\geq (p-1)u^{p-2}A^2 + (q-1)u^{2-p}B^2 \\ &= \tau A^2 + \tau^{-1}B^2 \end{aligned}$$

with  $\tau = (p-1)u^{p-2}$ .

Next we address the case  $u^p \leq v^q$ . This time  $\phi = u^p + v^q + \delta u^2 v^{2-q}$  and so

$$F_{u,v}(A, a, B, b) = a^2U + 2abV - b^2W + Z,$$

where

$$\begin{aligned} U &= p(p-2)u^{p-2}A^2 \\ V &= 2\delta(2-q)uv^{1-q}AB \\ W &= (2-q)q(\delta u^2 v^{-q} + v^{q-2})B^2 \\ Z &= (pu^{p-2} + 2\delta v^{2-q})A^2 + [(2-q)\delta u^2 v^{-q} + qv^{q-2}]B^2. \end{aligned}$$

These terms are all positive.

First let us fix  $u, v$  and minimize  $F_{u,v}(A, a, B, b)$  over all  $a, b \in [-1, 1]$ . Since  $F_{u,v}(A, -a, B, -b) = F_{u,v}(A, a, B, b)$ , we may assume that  $a \in [0, 1]$ . Then  $F_{u,v}(A, a, B, -b) \leq F_{u,v}(A, a, B, b)$  for non-negative  $b$ , since  $V > 0$ . Thus we can furthermore restrict ourselves to  $b \in [-1, 0]$ . But this is the same as minimizing  $\tilde{F}(a, b) = a^2U - 2abV - b^2W + Z$  over  $a, b \in [0, 1]$ .

The only stationary point of  $\tilde{F}$  is  $(0, 0)$ , which is obviously not the minimum. As for the boundary of  $[0, 1]^2$ , we quickly see there are two possibilities: if  $U \leq V$  then the minimum is attained at the point  $(1, 1)$  and

has the value  $U - 2V - W + Z$ , and if  $U \geq V$  then the minimum occurs at  $(V/U, 1)$  and takes the value  $-V^2/U - W + Z$ . In both cases the minimum is minorized by the expression  $Z - 2V - W$ . Now

$$\begin{aligned} Z - 2V - W &= (pu^{p-2} + 2\delta v^{2-q})A^2 - 4\delta(2-q)uv^{1-q}AB \\ &\quad + \delta[8v^{q-2} - (2-q)(q-1)u^2v^{-q}]B^2. \end{aligned}$$

Recall we are working under the assumption  $u^p \leq v^q$ . This gives estimates  $u^2v^{-q} \leq v^{q-2}$  and  $uv^{1-q} \leq 1$ . Consequently

$$Z - 2V - W \geq \delta(2v^{2-q}A^2 + 7v^{q-2}B^2) - 4\delta AB. \quad (9)$$

For any positive  $\lambda$  and  $v$  we can estimate

$$4AB \leq 2(\lambda v^{2-q}A^2 + \lambda^{-1}v^{q-2}B^2).$$

Together with (9) this implies

$$Z - 2V - W \geq \delta[2(1-\lambda)v^{2-q}A^2 + (7-2\lambda^{-1})v^{q-2}B^2].$$

By choosing, for example,  $\lambda = 1/2$  we get

$$Z - 2V - W \geq \delta(v^{2-q}A^2 + 3v^{q-2}B^2) \geq \delta(\tau A^2 + \tau^{-1}B^2)$$

where  $\tau = v^{2-q}$ .

Now we turn towards proving (iii). To estimate  $Q(v) - v \cdot \nabla Q(v)$  from below we need to estimate  $V \cdot \nabla \Phi(V) - \Phi(V)$  from below, where  $V := (\zeta, \eta)$  is now understood as a real  $(2M + 2N)$ -vector. We have

$$\nabla \Phi(V) = \frac{\partial \phi}{\partial x}(|\zeta|, |\eta|)\nabla|\zeta| + \frac{\partial \phi}{\partial y}(|\zeta|, |\eta|)\nabla|\eta|,$$

$$\zeta \cdot \nabla|\zeta| = \zeta \cdot e_\zeta = |\zeta| \quad \text{and} \quad \eta \cdot \nabla|\eta| = \eta \cdot e_\eta = |\eta|.$$

Combining this and writing again  $u = |\zeta|$ ,  $v = |\eta|$ , we get

$$V \cdot \nabla \Phi(V) - \Phi(V) = \frac{\partial \phi}{\partial u}(u, v)u + \frac{\partial \phi}{\partial v}(u, v)v - \phi(u, v).$$

Denote this expression by  $\Lambda(u, v)$ .

By now, having finished the proof of part (ii), we already have candidates for  $\tau$  we must work with.

Suppose  $u^p \geq v^q$ . Then a direct calculation shows that

$$\begin{aligned}\Lambda(u, v) &= \left(p - 1 + \frac{q - 1}{4}\right)u^p + (q - 1)\left(1 + \frac{(2 - q)(q - 1)}{8}\right)v^q \\ &\geq (p - 1)u^{p-2}u^2 + (q - 1)v^{q-2}v^2.\end{aligned}$$

But  $u^p \geq v^q$  implies  $v^{q-2} \geq u^{2-p}$ , which proves  $\Lambda(u, v) \geq \tau u^2 + \tau^{-1} v^2$  with the same  $\tau$  as in the corresponding case of (ii), namely,  $\tau = (p - 1)u^{p-2}$ .

Finally, suppose  $u^p \leq v^q$ . Then

$$\Lambda(u, v) = (p - 1)u^p + (q - 1)v^q + (3 - q)\delta u^2 v^{2-q}.$$

The first term on the right we simply drop out. Since  $q - 1 > \delta$  and  $3 - q \geq 1$  we get

$$\Lambda(u, v) \geq \delta(v^{q-2}v^2 + v^{2-q}u^2)$$

so the required inequality is proven with  $\tau = v^{2-q}$ , exactly as in the proof of (ii).  $\square$

**Remark.** In the proof we saw that when  $|\zeta|^p \geq |\eta|^q$  we get the properties (ii) and (iii) without  $\delta$ .

We also need to know how the gradient of  $Q$  behaves. Calculations carried out on the basis of (6) give estimates

$$\left|\frac{\partial Q}{\partial \zeta}\right| \leq C(p) \max\{|\zeta|^{p-1}, |\eta|\} \quad \text{and} \quad \left|\frac{\partial Q}{\partial \eta}\right| \leq C|\eta|^{q-1}. \quad (10)$$

Here of course

$$\frac{\partial Q}{\partial \zeta} = \left(\frac{\partial Q}{\partial \zeta_1}, \dots, \frac{\partial Q}{\partial \zeta_M}\right) \quad \text{and} \quad \frac{\partial Q}{\partial \eta} = \left(\frac{\partial Q}{\partial \eta_1}, \dots, \frac{\partial Q}{\partial \eta_N}\right).$$

Same estimates apply to the  $\bar{\partial}$ -derivatives of  $\zeta$  and  $\eta$ , for  $Q$  is a real-valued function.

### 3 Bilinear embedding

This section is devoted to proving Theorem 1. At this point it might be worth explaining the origins of the term “bilinear embedding” which we use throughout the article.

Let  $\phi$  be a  $C^1$  complex-valued function, defined on  $\mathbb{R}^n \times (0, \infty)$ . Write

$$\nabla_* \phi(x, t) = (\nabla \phi(x, t), x \phi(x, t)) \in \mathbb{C}^{2n+1}.$$

Then the statement of Theorem 1 implies the following one:

*The pairing  $\Psi$ , given by*

$$\Psi(f, g)(x, t) = \nabla_* P_t f(x) \cdot \nabla_* P_t g(x),$$

*defines a bounded bilinear mapping  $L^p \times L^q \rightarrow L^1(\mathbb{R}^n \times (0, \infty), dx dt)$  for all  $p \in (1, \infty)$ . Its norm is controlled by  $C(p^* - 1)$ .*

### 3.1 Structure of the proofs

We already emphasized that one of the main features of this presentation is the uniformity of the proofs regardless of the semigroup we work with. Having already introduced the Bellman function and its properties, we now wish to illustrate our strategy a bit further.

Given test functions  $f, g$  on  $\mathbb{R}^n$ , we want to define

$$v(x, t) := (P_t f(x), P_t g(x), P_t |f|^p(x), P_t |g|^q(x)) \quad (11)$$

and furthermore  $b := Q \circ v$ , that is,

$$b(x, t) := Q(P_t f(x), P_t g(x), P_t |f|^p(x), P_t |g|^q(x)). \quad (12)$$

For that purpose we have to check two things. One is that  $P_t |f|^p$  is well defined. The other is that  $v(x, t) \in \Omega$ . This is true for most of the natural semigroup extensions in which one can express  $P_t \varphi(x)$  as an integral of  $\varphi$  against some finite measure depending on  $(x, t)$  (see [5] for classical and Gaussian case and [30] for Hermite operator). This permits the inequality  $|P_t \varphi|^p \leq P_t |\varphi|^p$ , which is exactly what we need. Having explicit formulas at our disposal will settle the questions of well-posedness of  $b$  for our purposes. In general, though, it is not known to us for what class of  $\sqrt{L}$ -generated extensions this holds.

The main two steps in the proofs of the embedding theorems are always the same:

- Consider the operator

$$L' = \frac{\partial^2}{\partial t^2} - L.$$

It can be regarded as an extension of  $-L$  in the upper half-space. This extension is the “right one”, in the sense that  $L' \tilde{\varphi} = 0$ . Namely, this enables us to express  $L'b$  in terms of the Hessian of  $Q$ , and thereupon everything is set for applying the concavity properties (ii) and (iii) of  $Q$ .

- Our aim is to estimate the integral

$$- \int L'b(x, t) dx t dt \quad (13)$$

from below and above. The size property (i) of  $Q$  makes up for the upper estimate of the integral above, whereas (ii) and (iii) provide the estimates from below. The expressions which appear in the lower and the upper estimate of (13) are exactly those from the embedding theorem. That is, a more complete formulation of Theorem 1 would incorporate (13) as the middle term in the inequalities.

We carry out the plan described above. The particularity of the Hermite case is hidden in the fact that the said operator has a potential,  $|x|^2$ , which has to be reckoned with. This means that, in contrast with the situations studied in [5], the formula for  $L'b$  contains not only the scalar products involving the Hessian of  $Q$ , but also some other terms. However, the property (iii) of  $Q$  gives control exactly over these, newly arisen terms.

The proof of the Lemma below makes this statement more transparent.

### 3.2 Estimate of the integral (13) from below

The lower estimates of the integral (13) will trivially follow from the lower pointwise estimates of  $L'b(x, t)$  which we present next. Recall the notation  $\delta = q(q - 1)/8$ .

**Lemma 1.** *There is an absolute  $C > 0$  such that for all test functions  $f, g$  and all  $p \geq 2$  we have*

$$-L'b(x, t) \geq \delta \|P_t f(x)\|_* \|P_t g(x)\|_*. \quad (14)$$

**Proof.** By applying the chain rule we get

$$-L'b(x, t) = \sum_{j=0}^n \left\langle -d^2 Q(v_0) \frac{\partial v}{\partial x_j}(x, t), \frac{\partial v}{\partial x_j}(x, t) \right\rangle + |x|^2 [Q(v_0) - v_0 \cdot \nabla Q(v_0)]. \quad (15)$$

Here we wrote  $v_0 = v(x, t)$  and when  $j = 0$  we meant the differentiation in  $t$ . Combining (15) and Theorem 3 we find  $\tau = \tau(x, t) > 0$  such that

$$\begin{aligned} -L'b(x, t) &\geq \delta \tau \left( \sum_{j=0}^n \left| \frac{\partial}{\partial x_j} P_t f(x) \right|^2 + |x|^2 |P_t f(x)|^2 \right) \\ &\quad + \delta \tau^{-1} \left( \sum_{j=0}^n \left| \frac{\partial}{\partial x_j} P_t g(x) \right|^2 + |x|^2 |P_t g(x)|^2 \right). \end{aligned}$$

Now the inequality between the arithmetic and the geometric mean gives (14).  $\square$

**Remark.** Notice that if  $|x|^2$  is replaced by some arbitrary non-negative measurable function  $V(x)$  in the definition of  $L$  (and consequently in  $L'$ ,  $P_t$  and  $\|\cdot\|_*$ ), the statement from Lemma 1 does not change at all.

### 3.3 Estimates of the semigroup kernels

The aim of this section is to compile some of the known estimates for the integral kernels of the heat and Poisson semigroup, respectively, and to adapt them to suit our purposes. The estimates we have in mind are pointwise estimates of the kernel and its derivatives. We will need them in the continuation of the proof of Theorem 1, that is, when giving upper estimates of the integral  $-\int L'b$  introduced in (13).

#### 3.3.1 Estimates of the Hermite heat kernel

From [30, 4.1.2] we have that

$$e^{-tL}\varphi(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} K_t(x, y) \varphi(y) dy, \quad (16)$$

where [30, 4.1.3]

$$K_t(x, y) = \frac{1}{(\sinh 2t)^{n/2}} \exp \left( -\frac{|x|^2 + |y|^2}{2} \coth 2t + \frac{\langle x, y \rangle}{\sinh 2t} \right). \quad (17)$$

Direct calculation shows that

$$K_t(x, y) \leq (2t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}, \quad (18)$$

so the Hermite heat kernel is majorized by the ordinary heat kernel.

By Lemma 4.3.1.(i) and Lemma 4.3.2.(ii) from [30] there exist positive constants  $C, a$  not depending on  $x, y, t$  such that for  $j \in \{1, \dots, n\}$  we have

$$\left| \frac{\partial}{\partial x_j} K_t(x, y) \right| \leq C t^{-\frac{n+1}{2}} e^{-\frac{a}{t}|x-y|^2}, \quad (19)$$

whereas Lemma 4.1.1 (i) from the same source gives

$$\left| \frac{\partial}{\partial t} K_t(x, y) \right| \leq C t^{-\frac{n}{2}-1} e^{-\frac{a}{t}|x-y|^2}. \quad (20)$$

These estimates are valid for all  $t > 0$ .

### 3.3.2 Estimates of the Hermite Poisson kernel

Most of the inequalities just encountered can be transferred to the Poisson semigroup by the following well-known subordination principle.

Denote

$$d\mu(s) = \frac{1}{\sqrt{\pi}} e^{-s} s^{-1/2} ds.$$

This is a probability measure on  $(0, \infty)$ . The integral equation

$$e^{-\alpha} = \int_0^\infty e^{-\frac{\alpha^2}{4s}} d\mu(s)$$

gives rise to the subordination formula

$$P_t \varphi(x) = \int_0^\infty e^{-\frac{t^2}{4s} L} \varphi(x) d\mu(s). \quad (21)$$

By the same symbol as the extension, namely  $P_t$ , we also denote the Hermite Poisson kernel. From (21) and (16) we get

$$P_t(x, y) = \frac{1}{(2\pi)^{n/2}} \int_0^\infty K_{\frac{t^2}{4s}}(x, y) d\mu(s). \quad (22)$$

Hence, by (18),

$$P_t(x, y) \leq \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \cdot \frac{t}{(|x - y|^2 + t^2)^{\frac{n+1}{2}}}. \quad (23)$$

In other words our rough estimate shows that  $P_t(x, y)$  is majorized by the classical Poisson kernel in  $\mathbb{R}^n$ .

In order to give estimates of the derivatives of  $P_t$ , combine (22) and (19) to deduce

$$\left| \frac{\partial}{\partial x_j} P_t(x, y) \right| \leq \frac{Ct}{(|x - y|^2 + t^2)^{\frac{n+2}{2}}}. \quad (24)$$

As for the estimate of the derivative in  $t$ , fix  $(x, y)$  and write  $\psi(u) := K_u(x, y)$ . Then

$$\frac{\partial}{\partial t} [K_{\lambda t^2}(x, y)] = 2\lambda t \psi'(\lambda t^2). \quad (25)$$

We can calculate  $\psi'$  from (17):

$$\psi'(u) = \psi(u) \left[ \frac{|x|^2 + |y|^2 - 2\langle x, y \rangle \cosh 2u}{\sinh^2 2u} - n \coth 2u \right]. \quad (26)$$

The estimate for  $\psi'$ , however, comes from (20). We apply it together with (22) and (25). The result is

$$\left| \frac{\partial}{\partial t} P_t(x, y) \right| \leq \frac{C}{(|x - y|^2 + t^2)^{\frac{n+1}{2}}}. \quad (27)$$

**Remark.** It follows from (18) that  $(2\pi)^{-n/2} K_t(x, y) dy$  is, for each  $x \in \mathbb{R}^n$ , a positive sub-probability measure on  $\mathbb{R}^n$  (later on, see (58), we precisely calculate its mass). Jensen's inequality implies from here the pointwise estimate  $|e^{-tL}\varphi(x)|^p \leq e^{-tL}|\varphi|^p(x)$  whenever  $p \geq 1$ . By (21), the same is true for the Poisson semigroup, i.e.  $|P_t\varphi(x)|^p \leq P_t|\varphi|^p(x)$ . This in retrospect justifies our definition of function  $b$ , i.e. vector  $v(x, t)$  really maps into the domain of the Bellman function  $Q$ ; see the discussion ensuing the definitions (11), (12).

### 3.4 Integration by parts

In this section our goal is to extract the “noncontributing” part of the integral (13).

**Lemma 2.** *Let  $f$  and  $g$  belong to  $C_c^\infty$  and let  $b$  be as in (12). For all  $t > 0$ ,*

$$- \int_{\mathbb{R}^n} L'b(x, t) dx = \int_{\mathbb{R}^n} \left( - \frac{\partial^2 b}{\partial t^2}(x, t) + |x|^2 b(x, t) \right) dx. \quad (28)$$

**Proof.** Write

$$I := \int_{\mathbb{R}^n} \Delta b(x, t) dx.$$

Then (28) translates into showing that  $I = 0$ . By symmetry it suffices to do that for  $\partial^2/\partial x_1^2$  in place of  $\Delta$ .

Take  $M > 0$  and denote  $R_M = [-M, M]^n$ . If  $x = (x_1, \dots, x_n)$ , denote temporarily  $x' = (x_2, \dots, x_n)$ , so that we can write  $x = (x_1, x')$ . Then

$$\begin{aligned} & \int_{R_M} \frac{\partial^2 b}{\partial x_1^2}(x, t) dx_1 \dots dx_n \\ &= \underbrace{\int_{-M}^M \dots \int_{-M}^M}_{n-1} \left[ \frac{\partial b}{\partial x_1}(M, x', t) - \frac{\partial b}{\partial x_1}(-M, x', t) \right] dx_2 \dots dx_n. \end{aligned}$$

Thus

$$\left| \int_{R_M} \frac{\partial^2 b}{\partial x_1^2}(x, t) dx_1 \dots dx_n \right| \leq 2S (2M)^{n-1}, \quad (29)$$



where

$$S = \sup_{x \in \partial R_M} \left| \frac{\partial b}{\partial x_1}(x, t) \right|.$$

We want to estimate  $S$ . As in [5],

$$\frac{\partial b}{\partial x_1}(x, t) = \left\langle \nabla Q(v_0), \frac{\partial v}{\partial x_1}(x, t) \right\rangle,$$

where  $v_0 = v(x, t)$ . This means

$$\begin{aligned} \frac{\partial b}{\partial x_1}(x, t) = & \left\langle \frac{\partial Q}{\partial \zeta}(v_0), \frac{\partial}{\partial x_1} P_t f(x) \right\rangle_{\mathbb{C}^M} + \left\langle \frac{\partial Q}{\partial \eta}(v_0), \frac{\partial}{\partial x_1} P_t g(x) \right\rangle_{\mathbb{C}^N} \\ & + \left\langle \frac{\partial Q}{\partial \bar{\zeta}}(v_0), \frac{\partial}{\partial x_1} \overline{P_t f(x)} \right\rangle_{\mathbb{C}^M} + \left\langle \frac{\partial Q}{\partial \bar{\eta}}(v_0), \frac{\partial}{\partial x_1} \overline{P_t g(x)} \right\rangle_{\mathbb{C}^N} \\ & + \frac{\partial Q}{\partial Z}(v_0) \frac{\partial}{\partial x_1} P_t |f|^p(x) + \frac{\partial Q}{\partial H}(v_0) \frac{\partial}{\partial x_1} P_t |g|^q(x). \end{aligned} \quad (30)$$

Since in our case  $\zeta = P_t f(x)$  and  $\eta = P_t g(x)$ , (10) implies

$$\begin{aligned} \left| \frac{\partial b}{\partial x_1}(x, t) \right| \leq & C \left( (|P_t f(x)|^{p-1} + |P_t g(x)|) \left| \frac{\partial}{\partial x_1} P_t f(x) \right| \right. \\ & \left. + |P_t g(x)|^{q-1} \left| \frac{\partial}{\partial x_1} P_t g(x) \right| + \left| \frac{\partial}{\partial x_1} P_t |f|^p(x) \right| + \left| \frac{\partial}{\partial x_1} P_t |g|^q(x) \right| \right). \end{aligned} \quad (31)$$

Hence the estimation of  $S$  is reduced to estimating  $P_t \varphi(x)$  and  $\frac{\partial}{\partial x_1} P_t \varphi(x)$  with  $f, g, |f|^p, |g|^q$  in place of  $\varphi$ . In order to do that we recall the estimates from section 3.3.2.

Let the radius  $A > 0$  be such that the ball  $B(0, A)$  contains  $\text{supp } \varphi$ . If  $y \in \text{supp } \varphi \subseteq B(0, A)$  and  $|x| \geq 2A$  we get  $|x - y| \geq |x|/2$ . Then  $P_t \varphi(x) = \int_{B(0, A)} P_t(x, y) \varphi(y) dy$  and so for  $|x| \geq 2A$  the inequality (23) implies

$$|P_t \varphi(x)| \leq \frac{C(n)t \|\varphi\|_1}{|x|^{n+1}}. \quad (32)$$

Now we turn to the estimate of  $\frac{\partial}{\partial x_1} P_t \varphi(x)$ . First,

$$\left| \frac{\partial}{\partial x_1} P_t \varphi(x) \right| \leq \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_1} P_t(x, y) \right| |\varphi(y)| dy.$$

The inequality (24) implies

$$\left| \frac{\partial}{\partial x_1} P_t \varphi(x) \right| \leq \frac{C(n)t \|\varphi\|_1}{(|x|^2 + t^2)^{\frac{n+2}{2}}} \quad (33)$$

for sufficiently large  $x$  as specified above.

Note that  $x \in \partial R_M$  implies  $|x| \geq M$ . Now a combination of (29), (31), (32) and (33) shows that

$$\lim_{M \rightarrow \infty} \int_{R_M} \frac{\partial^2 b}{\partial x_1^2}(x, t) dx_1 \dots dx_n = 0.$$

This proves (28) and thus Lemma 2.  $\square$

### 3.5 Estimate of the integral (13) from above

Here we treat a consequence of Lemma 2 which consists of showing that the expression in (28), and for that matter the integral (13), are bounded by  $C(\|f\|_p^p + \|g\|_q^q)$ .

**Proposition 1.** *For all  $p > 1$ ,*

$$-\int_0^\infty \int_{\mathbb{R}^n} L' b(x, t) t dx dt \leq 6(\|f\|_p^p + \|g\|_q^q).$$

**Proof.** It clearly suffices to consider the case  $p \geq 2$ . By Lemma 2 we are done once we prove

$$\int |x|^2 b(x, t) dx t dt \leq 4(\|f\|_p^p + \|g\|_q^q) \quad (34)$$

and

$$-\int \frac{\partial^2 b}{\partial t^2}(x, t) dx t dt \leq 2(\|f\|_p^p + \|g\|_q^q). \quad (35)$$

#### 3.5.1 Proof of (34).

It follows from (i) on page 7 that

$$\int |x|^2 b(x, t) dx t dt \leq 2 \int |x|^2 (P_t |f|^p(x) + P_t |g|^q(x)) dx t dt. \quad (36)$$

The combination of (21), (16) gives

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^n} |x|^2 P_t |f|^p(x) dx t dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} |x|^2 \int_0^\infty e^{-\frac{t^2}{4s} L} |f|^p(x) d\mu(s) dx t dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} |x|^2 \int_0^\infty \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} K_{\frac{t^2}{4s}}(x, y) |f(y)|^p dy d\mu(s) dx t dt. \end{aligned}$$

It should suffice for our purpose to know that the integrals

$$\frac{1}{(2\pi)^{n/2}} \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty |x|^2 K_{\frac{t^2}{4s}}(x, y) d\mu(s) dx t dt \quad (37)$$

are bounded uniformly in  $y$  and  $n$ . By (17), the expression in (37) equals

$$\begin{aligned} & \frac{1}{(2\pi)^{n/2}} \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \frac{|x|^2}{(\sinh \frac{t^2}{2s})^{n/2}} \\ & \cdot \exp\left(-\frac{|x|^2 + |y|^2}{2} \coth \frac{t^2}{2s} + \frac{\langle x, y \rangle}{\sinh \frac{t^2}{2s}}\right) d\mu(s) dx t dt. \end{aligned}$$

First we integrate in  $x$ . Write temporarily  $\alpha = \coth \frac{t^2}{2s}$ ,  $\beta = \sinh \frac{t^2}{2s}$ . Consider

$$\begin{aligned} & \int_{\mathbb{R}^n} |x|^2 \exp\left(-\frac{|x|^2 + |y|^2}{2} \alpha + \frac{\langle x, y \rangle}{\beta}\right) dx \\ & = \int_{\mathbb{R}^n} \sum_{j=1}^n x_j^2 \exp\left(-\sum_{k=1}^n \left(\frac{x_k^2 + y_k^2}{2} \alpha - \frac{x_k y_k}{\beta}\right)\right) dx. \quad (38) \end{aligned}$$

Note that

$$\frac{\alpha}{2}(x_k^2 + y_k^2) - \frac{x_k y_k}{\beta} = \frac{1}{2} \left( \sqrt{\alpha} x_k - \frac{y_k}{\sqrt{\alpha\beta}} \right)^2 + \frac{y_k^2}{2\alpha},$$

hence we can continue (38) as

$$e^{-\frac{|y|^2}{2\alpha}} \sum_{j=1}^n \int_{\mathbb{R}^n} x_j^2 \prod_{k=1}^n e^{-\frac{1}{2} \left( \sqrt{\alpha} x_k - \frac{y_k}{\sqrt{\alpha\beta}} \right)^2} dx_1 \dots dx_n. \quad (39)$$

Now use that

$$\int_{\mathbb{R}} x^2 e^{-\frac{1}{2} \left( \sqrt{\alpha} x - \frac{y}{\sqrt{\alpha\beta}} \right)^2} dx = \sqrt{\frac{2\pi}{\alpha}} \left( \frac{y^2}{\alpha^2 \beta^2} + \frac{1}{\alpha} \right)$$

and

$$\int_{\mathbb{R}} e^{-\frac{1}{2} \left( \sqrt{\alpha} x - \frac{y}{\sqrt{\alpha\beta}} \right)^2} dx = \sqrt{\frac{2\pi}{\alpha}},$$

which simplifies (39) to

$$e^{-\frac{|y|^2}{2\alpha}} \left( \frac{2\pi}{\alpha} \right)^{n/2} \left( \frac{|y|^2}{\alpha^2 \beta^2} + \frac{n}{\alpha} \right).$$

Therefore we proved that the integral in (37) is equal to

$$\frac{1}{(2\pi)^{n/2}} \int_0^\infty \int_0^\infty \frac{1}{\beta^{n/2}} e^{-\frac{|y|^2}{2\alpha}} \left(\frac{2\pi}{\alpha}\right)^{n/2} \left(\frac{|y|^2}{\alpha^2\beta^2} + \frac{n}{\alpha}\right) d\mu(s) t dt$$

or

$$\int_0^\infty \int_0^\infty \frac{e^{-\frac{|y|^2}{2}} \tanh \frac{t^2}{2s}}{\cosh^{1+n/2} \frac{t^2}{2s}} \left(\frac{|y|^2}{\cosh \frac{t^2}{2s}} + n \sinh \frac{t^2}{2s}\right) d\mu(s) t dt. \quad (40)$$

Introduce a new variable  $u = t^2/2$  and write (40) as  $I_1 + I_2$ , where

$$\begin{aligned} I_1 &= \int_0^\infty \int_0^\infty |y|^2 e^{-\frac{|y|^2}{2} \tanh(u/s)} \cosh^{-2-n/2}(u/s) d\mu(s) du \\ I_2 &= \int_0^\infty \int_0^\infty e^{-\frac{|y|^2}{2} \tanh(u/s)} n \cosh^{-1-n/2}(u/s) \sinh(u/s) d\mu(s) du. \end{aligned}$$

Let us estimate  $I_1$  first. Obviously

$$I_1 \leq \int_0^\infty \int_0^\infty |y|^2 e^{-\frac{|y|^2}{2} \tanh(u/s)} \cosh^{-2}(u/s) du d\mu(s).$$

In the inner integral introduce a new variable  $w = e^{-\frac{|y|^2}{2} \tanh(u/s)}$ , from where we continue with

$$\begin{aligned} &= \int_0^\infty 2s \int_{e^{-\frac{|y|^2}{2}}}^1 dw d\mu(s) = 2 \left(1 - e^{-\frac{|y|^2}{2}}\right) \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s} s^{1/2} ds \\ &= 1 - e^{-\frac{|y|^2}{2}}. \end{aligned}$$

As for  $I_2$ , we have

$$I_2 \leq n \int_0^\infty \int_0^\infty \cosh^{-1-n/2}(u/s) \sinh(u/s) du d\mu(s).$$

This time we take  $w = \cosh^{-n/2}(u/s)$ , which gives

$$\int_0^\infty 2s \int_0^1 dw d\mu(s) = 1.$$

So we showed that

$$I_1 + I_2 \leq 2 - e^{-\frac{|y|^2}{2}} < 2.$$

Consequently

$$\int |x|^2 (P_t |f|^p(x) + P_t |g|^q(x)) dx t dt \leq 2(\|f\|_p^p + \|g\|_q^q),$$

which, in view of (36), implies (34).

**Remark.** This time we had to work with the exact formula for the kernels  $K_t$ ; the approximation (18) seems already a step to far, one where too much information is lost already.

### 3.5.2 Proof of (35)

It remains to prove a similar bound for the integral in (35). We follow the considerations from [5].

Integration by parts gives

$$\begin{aligned} & - \limsup_{\substack{\omega_1 \rightarrow 0 \\ \omega_2 \rightarrow \infty}} \int_{\mathbb{R}^n} \int_{\omega_1}^{\omega_2} \frac{\partial^2 b}{\partial t^2}(x, t) t \, dt \, dx = \underbrace{- \limsup_{t \rightarrow \infty} \int_{\mathbb{R}^n} t \frac{\partial b}{\partial t}(x, t) \, dx}_{\text{I}} \\ & + \underbrace{\liminf_{t \rightarrow 0} \int_{\mathbb{R}^n} t \frac{\partial b}{\partial t}(x, t) \, dx}_{\text{II}} + \underbrace{\liminf_{t \rightarrow \infty} \int_{\mathbb{R}^n} b(x, t) \, dx}_{\text{III}} - \underbrace{\limsup_{t \rightarrow 0} \int_{\mathbb{R}^n} b(x, t) \, dx}_{\text{IV}} . \end{aligned}$$

Let us start with III. By (12) and the property (i) of  $Q$  we have

$$\begin{aligned} \liminf_{t \rightarrow \infty} \int_{\mathbb{R}^n} b(x, t) \, dx & \leq 2 \liminf_{t \rightarrow \infty} \int_{\mathbb{R}^n} (P_t |f|^p(x) + P_t |g|^q(x)) \, dx \\ & \leq 2 \int_{\mathbb{R}^n} (|f|^p(x) + |g|^q(x)) \, dx , \end{aligned}$$

the second inequality following from the contractivity property of  $P_t$ .

Since function  $Q$  (and therefore  $b$ ) is positive, then so is the term IV, therefore we can skip it from all the estimates from above.

We are left with II and I. We want to show that they can be neglected. More precisely, we will prove that II  $\leq 0$  and I  $= 0$ .

In the estimates of II and I we will continuously be applying and referring to the decomposition of  $\frac{\partial b}{\partial t}(x, t)$  as in (30), just with  $t$  in place of  $x_1$ . This essentially gives four terms (nominally there are six of them, but  $\partial_{\zeta}$  and  $\partial_{\bar{\zeta}}$  derivatives are handled in the same fashion; likewise for  $\partial_{\eta}$  and  $\partial_{\bar{\eta}}$ ).

First we treat the terms in II and I corresponding, in the sense of (30) explained above, to the partial derivatives with respect to  $Z$  and  $H$ . These

derivatives are identically equal to 2, hence for the corresponding terms the estimate reduces to showing that, for  $\varphi = |f|^p + |g|^q$ ,

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial t} P_t \varphi(x) dx \leq 0, \quad (41)$$

which will imply that  $\underline{\mathbb{I}} \leq 0$  (more precisely, its part associated with the  $Z$ - and  $H$ - derivatives, see (30) on p. 17), and

$$\lim_{t \rightarrow \infty} t \int_{\mathbb{R}^n} \frac{\partial}{\partial t} P_t \varphi(x) dx = 0, \quad (42)$$

which will in turn imply that the “ $Z$ - and  $H$ - part” of  $\underline{\mathbb{I}}$  vanishes. Proving that is not as straightforward as in [5], since we cannot use the scalar product (i.e. duality) argument. Instead, we resort once again to the explicit formulas for the kernels. We start with the calculation of the integral appearing in (41) and (42).

By (21) and (16),

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial t} P_t \varphi(x) dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} \frac{\partial}{\partial t} K_{\frac{t^2}{4s}}(x, y) \varphi(y) dy d\mu(s) dx. \quad (43)$$

First we integrate with respect to  $x$ , i.e. we compute

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial t} K_{\frac{t^2}{4s}}(x, y) dx. \quad (44)$$

The formulæ (25) and (26) show that the integral in (44) can be written as

$$\frac{t}{2s\alpha^2} \int_{\mathbb{R}^n} K_{\frac{t^2}{4s}}(x, y) (|x|^2 + |y|^2 - 2\langle x, y \rangle \beta) dx - \frac{nt\beta}{2s\alpha} \int_{\mathbb{R}^n} K_{\frac{t^2}{4s}}(x, y) dx,$$

where now  $\alpha = \sinh \frac{t^2}{2s}$  and  $\beta = \cosh \frac{t^2}{2s}$ .

A computation shows this is the same as

$$\frac{t}{2s\alpha^2} e^{-\frac{\alpha|y|^2}{2\beta}} \left( \frac{2\pi}{\beta} \right)^{n/2} \left[ -\frac{\alpha^2}{\beta^2} |y|^2 + \frac{n\alpha}{\beta} \right] - \frac{nt\beta}{2s\alpha} e^{-\frac{\alpha|y|^2}{2\beta}} \left( \frac{2\pi}{\beta} \right)^{n/2},$$

thus

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial t} K_{\frac{t^2}{4s}}(x, y) dx = -\frac{t}{2s} e^{-\frac{\alpha|y|^2}{2\beta}} \left( \frac{2\pi}{\beta} \right)^{n/2} \left[ \frac{|y|^2}{\beta^2} + \frac{n\alpha}{\beta} \right]. \quad (45)$$

So we have, by (43) and (45),

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial t} P_t \varphi(x) dx = - \int_0^\infty \frac{t}{2s\beta^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{\alpha|y|^2}{2\beta}} \left[ \frac{|y|^2}{\beta^2} + \frac{n\alpha}{\beta} \right] \varphi(y) dy d\mu(s). \quad (46)$$

Note that the integrand is almost identical to the one in (40). More significantly, the above expression is non-positive, because  $\varphi \geq 0$ . This immediately implies (41).

As for (42), first write  $\varphi(y) \leq \|\varphi\|_\infty$  (of course we may assume that  $\varphi$  is bounded). One can pass to polar coordinates in  $\mathbb{R}^n$  and explicitly calculate

$$\int_{\mathbb{R}^n} e^{-\frac{\alpha|y|^2}{2\beta}} \left[ \frac{|y|^2}{\beta^2} + \frac{n\alpha}{\beta} \right] dy = (2\pi)^{n/2} n \left( \frac{\beta}{\alpha} \right)^{1+n/2}.$$

Hence (46) gives

$$0 \leq -t \int_{\mathbb{R}^n} \frac{\partial}{\partial t} P_t \varphi(x) dx \leq \|\varphi\|_\infty (2\pi)^{n/2} n \int_0^\infty \frac{t^2}{2s} \cdot \frac{\beta}{\alpha^{\frac{n}{2}+1}} d\mu(s).$$

Now

$$\frac{\beta}{\alpha^{\frac{n}{2}+1}} = \frac{1 + \alpha^2}{\beta \alpha^{\frac{n}{2}+1}} = \frac{1}{\beta \alpha^{\frac{n}{2}+1}} + \frac{1}{\beta \alpha^{\frac{n}{2}-1}},$$

therefore

$$\int_0^\infty \frac{t^2}{2s} \cdot \frac{\beta}{\alpha^{\frac{n}{2}+1}} d\mu(s) \leq \int_0^\infty \left( \frac{2s}{t^2} \right)^{\frac{n}{2}+1} d\mu(s) + C \int_0^\infty \left( \frac{2s}{t^2} \right)^{\frac{n}{2}} d\mu(s), \quad (47)$$

which clearly converges to 0 as  $t \rightarrow \infty$ . This confirms (42).

To get the first integral on the right of (47) we simply used that  $\beta \geq \alpha \geq t^2/(2s) =: v$ . And for the second one, we estimated  $\beta \alpha^{\frac{n}{2}-1} \geq \beta^{\frac{1}{2}} \alpha^{\frac{n-1}{2}} \geq \sqrt{e^v v^{n-1}/2}$ , and used that  $v^3 e^{-v} \leq C$ .

We have not yet finished the estimates of  $\underline{\mathbf{I}}$  and  $\underline{\mathbf{II}}$ . We still need to consider the terms in (30) corresponding to partial derivatives of  $Q$  with respect to  $\zeta$  and  $\eta$ .

Let  $v = v(x, t)$  be as in (11). Recall that the partial derivatives of  $Q$  were estimated in (10). Therefore, to estimate  $\int_{\mathbb{R}^n} |\frac{\partial Q}{\partial \zeta}(v)| |t \frac{\partial P_t f}{\partial t}(x)| dx$  and  $\int_{\mathbb{R}^n} |\frac{\partial Q}{\partial \eta}(v)| |t \frac{\partial P_t g}{\partial t}(x)| dx$  we need to estimate

$$A := \int_{\mathbb{R}^n} \max((P_t |f|)^{p-1}, P_t |g|) \left| t \frac{\partial P_t f}{\partial t} \right| dx$$

and

$$B := \int_{\mathbb{R}^n} (P_t|g|)^{q-1} \left| t \frac{\partial P_t g}{\partial t} \right| dx.$$

Let us prove first that

$$\lim_{t \rightarrow \infty} A = 0, \quad \lim_{t \rightarrow \infty} B = 0. \quad (48)$$

To do that recall the estimate (23) of the Hermite Poisson kernel. It implies, for a function  $\varphi \in L^1$ ,

$$P_t|\varphi|(x) \leq \frac{C_1}{t^n} \|\varphi\|_{L^1(\mathbb{R}^n)}$$

uniformly in  $x \in \mathbb{R}^n$ .

Now we are ready to prove (48). The previous inequality together with (27) implies

$$\begin{aligned} A &= \int_{\mathbb{R}^n} \max((P_t|f|)^{p-1}, P_t|g|)(x) \cdot \left| t \frac{\partial P_t f}{\partial t}(x) \right| dx \\ &\leq \frac{C(n)}{t^n} (\|f\|_1^{p-1} + \|g\|_1) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{t}{(|x-y|^2 + t^2)^{\frac{n+1}{2}}} |f(y)| dy dx \\ &= \frac{C'(n)}{t^n} (\|g\|_1 + \|f\|_1^{p-1}) \|f\|_1. \end{aligned}$$

This obviously tends to 0 as  $t \rightarrow \infty$ . The same with  $\lim_{t \rightarrow \infty} B$ . So (48) is proved which means that  $\bar{\mathbb{I}}$  disappears.

We are left with the task of proving  $\lim_{t \rightarrow 0} A = 0$  and  $\lim_{t \rightarrow 0} B = 0$ . Let us estimate  $\lim_{t \rightarrow 0} B$ , for example. From (23) it follows that  $\|P_t|g|\|_\infty \leq C(n)\|g\|_\infty$ . Therefore,

$$B \leq C(n)\|g\|_\infty^{q-1} \int_{\mathbb{R}^n} \left| t \frac{\partial P_t g}{\partial t}(x) \right| dx.$$

In order to estimate the integral on the right, the formula (46) is not enough. Instead, let us denote

$$\Phi_n(x, y, t) = t \frac{\partial P_t(x, y)}{\partial t}.$$

This is the integral kernel of the operator

$$\Lambda_t : \varphi \mapsto t \frac{\partial P_t \varphi}{\partial t}. \quad (49)$$



From (22) and (25) we get

$$\Phi_n(x, y, t) = C(n) \int_0^\infty \psi'_{x,y}(u) \frac{t}{\sqrt{u}} e^{-\frac{t^2}{4u}} du, \quad (50)$$

where  $\psi_{x,y}(u) = K_u(x, y)$ , as on page 15. Now

$$\int_{\mathbb{R}^n} \left| t \frac{\partial P_t g}{\partial t}(x) \right| dx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\Phi_n(x, y, t)| |\varphi(y)| dy dx.$$

From (50) we see that the integrand on the right goes to zero pointwise as  $t \rightarrow 0$ . On the other hand, we can apply (20) to estimate  $|\psi'_{x,y}(u)|$  and, consequently, (50). The calculation which unfolds shows that, for all  $t > 0$ , the function  $(x, y) \rightarrow |\Phi_n(x, y, t)| |\varphi(y)|$  has a majorant from  $L^1(\mathbb{R}^n \times \mathbb{R}^n)$  which is independent on  $t$ . This means we are entitled to use the dominated convergence theorem which gives

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^n} \left| t \frac{\partial P_t g}{\partial t}(x) \right| dx = 0 \quad (51)$$

and so  $\lim_{t \rightarrow 0} B = 0$ . The same with  $A$ . We finally proved that  $\underline{\Pi} \leq 0$ .  $\square$

**Remark.** Recall that  $\Lambda_t$  were defined in (49). Thus (51) can be reformulated as  $\lim_{t \rightarrow \infty} \|\Lambda_t g\|_1 = 0$ . There is also an alternative way to prove (51). First notice the statement is trivial if  $g$  belongs to  $\mathcal{V}$ , the space of linear combinations of Hermite functions (in order to see the definition of the latter the reader is prompted to move to page 27). This follows from observing that  $\Lambda_t = t P_t \sqrt{L}$  and applying (53), since it implies that  $\sqrt{L}$  preserves  $\mathcal{V}$ . Now take arbitrary compactly supported  $g$  and  $\varepsilon > 0$ . By Lemma 3 we can find  $\tilde{g} \in \mathcal{V}$  such that  $\|g - \tilde{g}\|_1 < \varepsilon$ . Since, by (27), the kernels of  $\Lambda_t$  are majorized, up to some constant  $C$ , by the usual Poisson kernel, we have that  $\|\Lambda_t\|_{B(L^1)} \leq C$  uniformly in  $t$ . Thus  $\|\Lambda_t(g - \tilde{g})\|_1 < C\varepsilon$ . As  $\tilde{g} \in \mathcal{V}$ , there is  $\delta > 0$  such that for  $0 < t < \delta$  we have  $\|\Lambda_t \tilde{g}\|_1 < \varepsilon$ . Therefore for such  $t$  we conclude  $\|\Lambda_t g\|_1 < (C + 1)\varepsilon$ .

### 3.6 Proof of Theorem 1

Basically we are done already. Note that, for  $p \geq 2$ , Lemma 1 and Proposition 1 together give

$$\int_0^\infty \int_{\mathbb{R}^n} \|P_t f(x)\|_* \|P_t g(x)\|_* dx dt \leq C(p-1)(\|f\|_p^p + \|g\|_q^q)$$

with  $C = 48$ . The same inequality obviously also holds for  $1 < p \leq 2$  if instead of  $p - 1$  we take  $q - 1$ . Now replace  $f$  by  $\lambda f$  and  $g$  by  $\lambda^{-1}g$  whereupon take the minimum in  $\lambda > 0$ . While the left-hand side does not change, we get  $C(p - 1)p^{1/p}q^{1/q}\|f\|_p\|g\|_q$  on the right-hand side. Since  $p^{1/p}q^{1/q} \leq 2$ , we obtain the desired statement of Theorem 1.  $\square$

### 3.7 Schrödinger operators with positive potential

Following the steps in the proof of Theorem 1 we readily prove the next result.

**Theorem 4.** *Let  $V$  be a non-negative function on  $\mathbb{R}^n$  satisfying properties (a)–(d) from page 4. There is an absolute constant  $C > 0$  such that for arbitrary natural numbers  $M, N, n$ , any pair  $f : \mathbb{R}^n \rightarrow \mathbb{C}^M$  and  $g : \mathbb{R}^n \rightarrow \mathbb{C}^N$  of  $C_c^\infty$  test functions and any  $p > 1$  we have*

$$\int_0^\infty \int_{\mathbb{R}^n} \|P_t f(x)\|_* \|P_t g(x)\|_* dx dt \leq C(p^* - 1) \|f\|_p \|g\|_q.$$

The constant  $C$  only depends on the constant  $C_0$  from the property (d).

Let us outline the reasons why the properties (a)–(d) were needed. First, we want to make sure that the function  $v(x, t)$  really maps into the domain of our Bellman function. For that reason we need that  $|P_t \varphi(x)|^p \leq P_t |\varphi|^p(x)$ . This happens if the Poisson kernel of  $L$  defines a sub-probability measure at any level. But, owing to the subordination formula (21), it is enough to have that for the heat kernel associated to  $L$ . This is exactly property (a). Properties (b) and (c) are used to justify the estimate of the integral  $-\int L'b$  from above, see sections 3.4 and 3.5. Property (d) replaces section 3.5.1.

As for the lower estimates, it was already noted in a remark on page 14 that they are completely independent of the choice of potential  $V$ .

## 4 Riesz transforms

In this section we apply our embedding theorem to obtain estimates of Riesz transforms associated to Hermite operator. Let us first introduce the necessary objects.

*Hermite functions*  $h_m$ ,  $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , are on  $\mathbb{R}$  defined as

$$h_m(x) = c_m (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{-x^2}$$

taken with the  $L^2(\mathbb{R})$  normalization  $c_m = (2^m m! \sqrt{\pi})^{-1/2}$ .

If  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ , then the *Hermite function* on  $\mathbb{R}^n$  is given by

$$h_\alpha := h_{\alpha_1} \otimes \dots \otimes h_{\alpha_n}.$$

Next we provide some argumentation as to why it is convenient to take linear combinations of  $h_\alpha$ 's as the family of test functions.

**Lemma 3.** *The space  $\text{Lin} \{h_\alpha; \alpha \in \mathbb{N}_0^n\}$  is dense in  $L^p(\mathbb{R}^n)$  for  $1 \leq p < \infty$ .*

**Proof.** Take  $f \in C_c^\infty(\mathbb{R}^n)$  and define coefficients  $\hat{f}(\alpha)$  as

$$\hat{f}(\alpha) = \int_{\mathbb{R}^n} f(x) h_\alpha(x) dx.$$

Consider

$$S_N f := \sum_{|\alpha| \leq N} \hat{f}(\alpha) h_\alpha.$$

We can repeat the proof of Lemma 5.4.1 in [30] to show that  $S_N f$  converge to  $f$  in the  $L^p$  norm.  $\square$

**Remark.** The previous sentence is not true for arbitrary  $f \in L^p$ . In fact, a well-known theorem by Askey and Wainger states that already when  $n = 1$ , this is the case if and only if  $p^* < 4$ .

Recall that  $\mathcal{A}_j$  and  $\mathcal{A}_j^*$  were introduced in (2). By [30, 1.1.30],

$$\begin{aligned} \mathcal{A}_j h_\alpha &= \sqrt{2(\alpha_j + 1)} h_{\alpha_1} \otimes \dots \otimes h_{\alpha_j+1} \otimes \dots \otimes h_{\alpha_n} \\ &= \sqrt{2(\alpha_j + 1)} h_{\alpha+e_j} \end{aligned} \tag{52}$$

and similarly

$$\mathcal{A}_j^* h_\alpha = \sqrt{2\alpha_j} h_{\alpha-e_j}.$$

Together with (1) this implies

$$Lh_\alpha = (2|\alpha| + n)h_\alpha, \tag{53}$$

where  $|\alpha| = \alpha_1 + \dots + \alpha_n$ .

## 4.1 Spectral multipliers

Another tool we need in order to treat the Riesz transforms are spectral multipliers. They are defined as follows.

Let  $\Psi$  be a bounded complex function on  $\mathbb{N}$ . In view of (53) it is natural to define

$$\Psi(L) := \sum_{m=0}^{\infty} \Psi(2m+n) \mathcal{P}_m,$$

where  $\mathcal{P}_m$  is the projector onto the subspace of  $L^2(\mathbb{R}^n)$  generated by all Hermite functions  $h_\alpha$  with  $|\alpha| = m$ .

We are interested in  $L^p$  boundedness of such operators. The sheer boundedness of  $\Psi$  does not guarantee that (unless  $p = 2$ ). Certain sufficient conditions are given by the multiplier theorems for Hermite expansions due to Mauceri [16] and Thangavelu [30, Theorem 4.2.1]. They imposed Marcinkiewicz-Hörmander-Mihlin-type conditions on their multipliers. Results in the same spirit were also obtained for closely related operators such as Weyl multipliers [16], [28] and multipliers associated with the twisted Laplacian [18]. We should also mention the paper [7] which itself contains many further references. But for our purpose we need more – we want estimates independent of  $n$  and  $p$ . They are provided by Theorem 2 which is proven below. It basically confirms the assertion made in [5], see Remark 3.2 there, namely, that the method exposed in [5] is only dependent on successful treatment of the corresponding spectral multipliers. The latter, in turn, should follow from “non-singularity” of the spectrum of the underlying differential operator (in our case,  $L$ ).

**Proof of Theorem 2.** Let  $\Phi(z) = \Psi(1/z)$ . The assumption on  $\Psi$  can be restated to say that  $\Phi$  is analytic in a neighbourhood of 0. If  $R$  is the radius of convergence of its power series expansion around 0, let  $\rho = 1/R$ . Thus  $\Psi$  is analytic in  $\{|z| > \rho\}$ .

For the sake of convenience assume that  $\Phi(0) = 0$ ; it is trivial to remove this restriction. Therefore  $\Phi$  can be expanded as  $\Phi(w) = \sum_{j=1}^{\infty} c_j w^j$ , provided that  $|w| < R$ . The Cauchy-Hadamard formula gives

$$\rho = \limsup_{j \rightarrow \infty} |c_j|^{1/j}. \quad (54)$$

Suppose first that  $n > \rho$ . In that case we can write  $\Psi(L) = \Phi(L^{-1}) = \sum_{j=1}^{\infty} c_j L^{-j}$ . Choose  $a > \rho$  which is also strictly smaller than any integer

that exceeds  $\rho$ . In other words, fix  $a \in (\rho, [\rho] + 1)$ . We have the formula

$$L^{-j} = \frac{1}{(j-1)!} \int_0^\infty t^{j-1} e^{-tL} dt.$$

Then

$$\|L^{-j}\|_{p \rightarrow p} \leq \frac{1}{(j-1)!} \int_0^\infty t^{j-1} \|e^{-tL}\|_{p \rightarrow p} dt.$$

Since  $a > \rho$ , it follows from (54) that there is  $C = C(a) > 0$  such that  $|c_j| \leq Ca^j$  for all  $j \in \mathbb{N}$ . Consequently,

$$\|\Phi(L^{-1})\|_{p \rightarrow p} \leq Ca \int_0^\infty e^{at} \|e^{-tL}\|_{p \rightarrow p} dt. \quad (55)$$

Hence in order to proceed we must estimate the norm of  $e^{-tL}$  on  $L^p$ .

Heat semigroup  $e^{-tL}$  is known to be contractive in every  $L^p$ . Moreover, in  $L^2$  it is very contractive, in the sense that

$$\|e^{-tL}\|_{2 \rightarrow 2} = e^{-nt}, \quad (56)$$

just because the smallest eigenvalue of  $L$  is  $n$  if we are in a  $n$ -dimensional space.

On the other hand, we have by (16) that

$$\|e^{-tL}\|_{\infty \rightarrow \infty} \leq \frac{1}{(2\pi)^{n/2}} \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} K_t(x, y) dy, \quad (57)$$

where  $K_t$  is as in (17). By taking  $\varphi \equiv 1$  in (16) we see that we actually have an equality in (57). A calculation, very similar to the one from the first part of Section 3.5, shows that

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} K_t(x, y) dy = (\cosh 2t)^{-n/2} e^{-\frac{|x|^2}{2 \coth 2t}}, \quad (58)$$

hence we can continue (57) as

$$\|e^{-tL}\|_{\infty \rightarrow \infty} = (\cosh 2t)^{-n/2}. \quad (59)$$

Note that exactly the same identity is valid on  $L^1$ .

Complex interpolation between (56) and (59) yields, for arbitrary  $p \in [1, \infty]$ ,

$$\|e^{-tL}\|_{p \rightarrow p} \leq e^{-2nt/p^*} (\cosh 2t)^{-\frac{n}{2}\gamma(p)},$$

where  $\gamma(p) = 1 - 2/p^*$ . By applying it in (55) we get

$$\|\Phi(L^{-1})\|_{p \rightarrow p} \leq Ca \int_0^\infty e^{(a-n)t} [e^{-2t} \cosh 2t]^{-\frac{n}{2}\gamma(p)} dt. \quad (60)$$

This integral converges if and only if  $n > a$ , that is, if and only if  $n > \rho$ . In order to estimate it from above, we first estimate the cosh part of the integrand as follows:

$$\cosh 2t \geq \max\{1, e^{2t}/2\} = \begin{cases} 1 & ; \quad t \leq t_0 \\ e^{2t}/2 & ; \quad t \geq t_0 \end{cases}, \quad (61)$$

where  $t_0 = \log \sqrt{2}$ . Next we write the integral in (60) as

$$\int_0^\infty = \int_0^{t_0} + \int_{t_0}^\infty$$

and in each of the integrals on the right apply the appropriate estimate from (61). The resulting integrals can be explicitly calculated. One obtains

$$\frac{1}{Ca} \|\Phi(L^{-1})\|_{p \rightarrow p} \leq \frac{\sqrt{2}^\alpha - 1}{\alpha} + \frac{\sqrt{2}^\alpha}{\beta}, \quad (62)$$

where  $\alpha = a - 2n/p^*$  and  $\beta = n - a$ . This estimate is valid as long as  $\beta > 0$  (in the case of  $\alpha = 0$  one has the limiting expression, i.e.  $\log \sqrt{2} + 1/\beta$ ). The right-hand side of (62) is increasing in  $\alpha$ . But obviously  $\alpha < a$ , which means that the expression in (62) is uniformly bounded as  $p$  ranges over  $[1, \infty]$  and  $n$  ranges over integers bigger than  $\rho$ . The assumption  $n > \rho$  was used here for the second time, namely, to estimate  $\beta \geq [\rho] + 1 - a > 0$ .

Now let us consider the case when  $n \leq \rho$ . Assume first that  $n = 1$ . We are indebted to Adam Sikora for showing us how to proceed in this case.

We would like to show that the operator  $(L + I)^{-1/2}$  maps boundedly  $L^2 \rightarrow L^\infty$  (then, by duality, we also get  $L^1 \rightarrow L^2$  boundedness). Indeed, for a Schwarz function  $u$  we have

$$\begin{aligned} |u(x)| &\leq \int_{\mathbb{R}} |\widehat{u}| = \int_{\mathbb{R}} |\widehat{u}(x)| (1 + |x|^2)^{1/2} (1 + |x|^2)^{-1/2} dx \\ &\leq C \left( \int_{\mathbb{R}} |\widehat{u}(x)|^2 (1 + |x|^2) dx \right)^{1/2} = C \langle (|x|^2 + 1) \widehat{u}, \widehat{u} \rangle^{1/2} \\ &= C \langle [(-\Delta + I)u]^\wedge, \widehat{u} \rangle^{1/2} = C \langle (-\Delta + I)u, u \rangle^{1/2} \leq C \langle (L + I)u, u \rangle^{1/2}. \end{aligned} \quad (63)$$

Conversely, let us show that  $(L + I)^{-1/2}$  is also bounded as an operator  $L^\infty \rightarrow L^2$ . Since this is a self-adjoint operator, it suffices to show that it is bounded  $L^2 \rightarrow L^1$ . But this simply follows from the Hölder's inequality:

$$\|u\|_1^2 \leq C\|(|x|^2 + I)^{1/2}u\|_2^2 = C\langle(|x|^2 + I)u, u\rangle \leq C\langle(L + I)u, u\rangle. \quad (64)$$

At the beginning of the proof we assumed that  $\Phi(z) = 0$ . This quickly implies that

$$|\Psi(\lambda)| \leq \frac{C}{\lambda + 1} \quad (65)$$

for some  $C > 0$ . Therefore

$$\|\Psi(L)\|_{\infty \rightarrow \infty} \leq \|(L + I)^{-1/2}\|_{2 \rightarrow \infty} \|\Psi(L)(L + I)\|_{2 \rightarrow 2} \|(L + I)^{-1/2}\|_{\infty \rightarrow 2}.$$

Thus  $\Psi(L)$  is bounded  $L^\infty \rightarrow L^\infty$ . Now the Riesz-Thorin theorem implies that there is an absolute  $C > 0$ , such that  $\|\Psi(L)\|_{p \rightarrow p} \leq C$  for all  $p \geq 2$ . Since  $\Psi(L)^* = \overline{\Psi}(L)$ , and since  $\overline{\Psi}$  also satisfies (65), we get  $\|\Psi(L)\|_{p \rightarrow p} \leq C$  for  $1 < p \leq 2$ , as well, while boundedness on  $L^1$  can be proven directly as above.

Basically the same proof is valid for arbitrary  $n \in \mathbb{N}$ . The difference is that in general one needs to take the  $n/2$ -th power of  $x^2 + 1$  in order to run the Hölder's inequality in (64). So one should deal with  $(L + I)^n$ . But when  $n > 1$  one cannot, as in (63) and (64), simply discard the terms  $|x|^2$  or  $-\Delta$  in the lower estimate of  $\langle(L + I)^n u, u\rangle$ . However, the  $L^2 \rightarrow L^1$  boundedness of  $S := (L + I)^{-n/2}$  is for arbitrary  $n$  provided by the estimate (7.11) from [7]. To obtain the  $L^2 \rightarrow L^\infty$  boundedness we use the Fourier transform. It convenes us to define it on  $\mathbb{R}^n$  as

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i\langle x, \xi \rangle} dx.$$

For then  $\widehat{-\Delta f}(\xi) = |\xi|^2 \hat{f}(\xi)$  and  $\widehat{|x|^2 f}(\xi) = -\Delta \hat{f}(\xi)$ , thus  $\widehat{Lu} = L\hat{u}$ , i.e. the Fourier transform commutes with  $L$ . Consequently,

$$\|Su\|_\infty \leq \|\widehat{Su}\|_1 = \|S\hat{u}\|_1 \leq C\|\hat{u}\|_2 = C\|u\|_2.$$

Finally, these modifications call for a suitably sharper estimate in (65), namely  $|\Psi(\lambda)| \leq C(\lambda + 1)^{-n}$ , in order to repeat the calculation which follows it. But this can be easily achieved, since we may assume without loss of generality that the first  $n$  derivatives of  $\Phi$  at zero vanish.

All this shows that  $\|\Psi(L)\|_{B(L^p(\mathbb{R}^n))} \leq C(n)$ . Now just take maximum of the constant, which appeared in the estimates for  $n > \rho$ , and all  $C(n)$  for  $1 \leq n \leq \rho$ . This is our absolute constant.  $\square$

## 4.2 Proof of Corollaries 1 and 2

If viewed correctly, these are consequences of Theorems 1 and 2. The connection between them will be established through the following two formulas:

$$\langle R_j f, g \rangle = \int_0^\infty \left\langle \mathcal{A}_j P_t \mathcal{O} f, \frac{\partial}{\partial t} P_t g \right\rangle_{L^2(\mathbb{R}^n)} t dt \quad (66)$$

and

$$\langle R_j^* f, g \rangle = \int_0^\infty \left\langle \mathcal{A}_j^* P_t \mathcal{O}^* f, \frac{\partial}{\partial t} P_t g \right\rangle_{L^2(\mathbb{R}^n)} t dt. \quad (67)$$

Here  $\mathcal{O}$  and  $\mathcal{O}^*$  are operators in  $L^p(\mathbb{R}^n)$ ,  $1 < p < \infty$ , hopefully bounded independently of dimension  $n$ . In order to calculate these operators we test the formulas on Hermite functions. More precisely, take

$$f := L^{\frac{1}{2}} h_\alpha.$$

Then (66) becomes

$$\langle \mathcal{A}_j h_\alpha, g \rangle = - \int_0^\infty \left\langle L^{\frac{1}{2}} P_t \mathcal{A}_j P_t \mathcal{O} L^{\frac{1}{2}} h_\alpha, g \right\rangle t dt. \quad (68)$$

Write formally

$$\mathcal{O} h_\alpha = \sum_{\beta \in \mathbb{N}_0^n} o_{\alpha\beta} h_\beta. \quad (69)$$

By using (52), (53) and (69), we formally calculate

$$L^{\frac{1}{2}} P_t \mathcal{A}_j P_t \mathcal{O} L^{\frac{1}{2}} h_\alpha = \sqrt{2|\alpha| + n} \sum_{\beta \in \mathbb{N}_0^n} o_{\alpha\beta} e^{-t\sqrt{2|\beta|+n}} \lambda_{|\beta|+1}(t) \sqrt{2(\beta_j + 1)} h_{\beta+e_j}.$$

Here

$$\lambda_k(t) = \sqrt{2k + n} e^{-t\sqrt{2k+n}}$$

Together with (68) this means that we can take  $o_{\alpha\beta} = 0$  if  $\alpha \neq \beta$  and we have a formula for the coefficients  $o_{\alpha\alpha}$ , which we can denote by  $o_\alpha$ :

$$\frac{1}{o_\alpha} = - \int_0^\infty \lambda_{|\alpha|}(t) \lambda_{|\alpha|+1}(t) t dt.$$

Thus  $o_\alpha$  and  $o_\alpha^*$  actually depend on  $|\alpha|$  only, so if we denote  $m = |\alpha|$ , we may write  $o_\alpha = o_m$ ,  $o_\alpha^* = o_m^*$ . We get

$$o_m = - \frac{(\sqrt{2m+n} + \sqrt{2m+n+2})^2}{\sqrt{2m+n} \sqrt{2m+n+2}}, \quad o_m^* = - \frac{(\sqrt{2m+n} + \sqrt{2m+n-2})^2}{\sqrt{2m+n} \sqrt{2m+n-2}}$$



or, equivalently,

$$o_m = -\Psi(2m + n) \quad \text{and} \quad o_m^* = o_{m-1},$$

where

$$\Psi(k) = \frac{(\sqrt{k} + \sqrt{k+2})^2}{\sqrt{k}\sqrt{k+2}}. \quad (70)$$

Consequently,

$$\mathcal{O} = \sum_{m \in \mathbb{N}_0} o_m \mathcal{P}_m \quad \text{and} \quad \mathcal{O}^* = \sum_{m \in \mathbb{N}_0} o_m^* \mathcal{P}_m,$$

recalling that  $\mathcal{P}_m$  is the projector onto the subspace of  $L^2(\mathbb{R}^n)$  generated by all Hermite functions  $h_\alpha$  with  $|\alpha| = m$ .

Note that  $o_0^*$  is defined if  $n \geq 3$ . So if  $n = 1, 2$  we have to correct formula (67) as  $o_0^* := 0$  and

$$\langle R_j^* f, g \rangle = \langle R_j^* \mathcal{P}_0 f, g \rangle + \int_0^\infty \left\langle A_j^* P_t \mathcal{O}^* f, \frac{\partial}{\partial t} P_t g \right\rangle t \, dt.$$

But (53) implies that  $L^{\frac{1}{2}} h_0 = \sqrt{n} h_0$ , therefore

$$R_j^* \mathcal{P}_0 f = A_j^* L^{-\frac{1}{2}} \langle f, h_0 \rangle h_0 = \langle f, h_0 \rangle \frac{1}{\sqrt{n}} A_j^* h_0 = 0.$$

We actually proved that for any  $n$  we can take  $o_0^* = 0$  and (67) remains valid.

**Remark.** It does not come as a surprise that the formulæ for  $o_m, o_m^*$  are very similar to those in [5, p. 183]. See also Remark 3.2 in the same paper.

By applying Theorem 2 we immediately get the following result.

**Proposition 2.** *For all  $p \in [1, \infty]$  and all  $n \in \mathbb{N}$ , operators  $\mathcal{O}$  and  $\mathcal{O}^*$  are bounded on  $L^p(\mathbb{R}^n)$  with constants independent of  $n$  and  $p$ .*

**Proof.** Indeed,  $\mathcal{O} = \Psi(L)$  with  $\Psi$  as in (70), while  $\mathcal{O}^* = \Psi^*(L)$ , where  $\Psi^*(k) = \Psi(k-2)$  for  $k > 2$  and  $\Psi^*(1) = \Psi^*(2) = 0$ .  $\square$

Let us show how (66) and (67) help to estimate Riesz transforms. Take  $m \in \mathbb{N}$  and let  $f = (f_1, \dots, f_m)$  be a  $\mathbb{C}^m$ -valued test function on  $\mathbb{R}^n$ . By  $R_j f$  we will understand  $(R_j f_1, \dots, R_j f_m)$ ; similarly for  $R_j^*$ . Let also  $\mathcal{R}f = (R_1 f, \dots, R_n f, R_1^* f, \dots, R_n^* f)$ . This is a function with values in  $(\mathbb{C}^m)^{2n}$ .

Thus we can think of  $\mathcal{R}f$  as a matrix function with entries  $R_j f_k$  and  $R_j^* f_k$ , where  $j = 1, \dots, n$  and  $k = 1, \dots, m$ . Therefore

$$\|\mathcal{R}f\|_p^p = \int_{\mathbb{R}^n} \left( \sum_{j,k} |R_j f_k(x)|^2 + |R_j^* f_k(x)|^2 \right)^{p/2} dx = \int_{\mathbb{R}^n} \|\mathcal{R}f(x)\|_{HS}^p dx,$$

where  $\|\cdot\|_{HS}$  stands for the usual Hilbert-Schmidt norm.

**Proposition 3.** *Under the above notation,*

$$\|\mathcal{R}f\|_p \leq C(p^* - 1)\|f\|_p.$$

for some absolute  $C > 0$  and all  $n \in \mathbb{N}$ ,  $p > 1$ .

Note that Corollary 1 is just a special case (when  $m = 1$ ) of this proposition, while Corollary 2 follows by applying it repeatedly. Therefore it is only left for us to prove Proposition 3.

**Proof.** Observe that

$$\|\mathcal{R}f\|_p = \sup |\langle (R_1 f, \dots, R_n f, R_1^* f, \dots, R_n^* f), (g_1, \dots, g_{2n}) \rangle|,$$

the supremum being taken over all  $g = (g_1, \dots, g_{2n})$  with  $L^q$  norm (in the appropriate space) equal to one, where each  $g_j$  is a function  $\mathbb{R}^n \rightarrow \mathbb{C}^m$ .

Now by (66) and (67),

$$\begin{aligned} & |\langle (R_1 f, \dots, R_n f, R_1^* f, \dots, R_n^* f), (g_1, \dots, g_{2n}) \rangle| = \\ & \left| \int_0^\infty \int_{\mathbb{R}^n} \sum_{j=1}^n \left( \langle \mathcal{A}_j P_t \mathcal{O} f(x), \partial_t P_t g_j(x) \rangle_{\mathbb{C}^m} + \langle \mathcal{A}_j^* P_t \mathcal{O}^* f(x), \partial_t P_t g_{n+j}(x) \rangle_{\mathbb{C}^m} \right) dx t dt \right|. \end{aligned}$$

Here, as before, by  $\mathcal{O}f$  we mean  $(\mathcal{O}f_1, \dots, \mathcal{O}f_m)$ ; similarly for  $\mathcal{A}_j P_t$ ,  $\mathcal{A}_j^* P_t \mathcal{O}^*$  and  $\partial_t P_t$ . By the Cauchy-Schwarz inequality we get

$$\leq \int_0^\infty \int_{\mathbb{R}^n} \left( \sum_{j=1}^n |\mathcal{A}_j P_t \mathcal{O} f(x)|^2 + \sum_{j=1}^n |\mathcal{A}_j^* P_t \mathcal{O}^* f(x)|^2 \right)^{1/2} \left( \sum_{j=1}^{2n} |\partial_t P_t g_j(x)|^2 \right)^{1/2} dx t dt. \quad (71)$$

We continue by a raw estimate

$$\leq \sqrt{2} \int_0^\infty \int_{\mathbb{R}^n} \left( \|P_t \mathcal{O} f(x)\|_* + \|P_t \mathcal{O}^* f(x)\|_* \right) \|P_t g(x)\|_* dx t dt \quad (72)$$

whereupon Theorem 1 yields

$$\leq C(p^* - 1) \left( \|\mathcal{O}f\|_p + \|\mathcal{O}^* f\|_p \right) \|g\|_q. \quad (73)$$

In Proposition 2 we proved  $\|\mathcal{O}\|_{B(L^p(\mathbb{R}^n))} \leq C$  with some absolute  $C > 0$  and the same for  $\|\mathcal{O}^*\|_{B(L^p(\mathbb{R}^n))}$ . From the theorem of Marcinkiewicz and Zygmund (see, for example, [10]) it follows that the same bounds apply to the  $l^2$ -valued extensions of  $\mathcal{O}$  and  $\mathcal{O}^*$  that appear in (73). Therefore the proof of Proposition 3 (and consequently of Corollaries 1 and 2) is complete.  $\square$

### 4.3 Heisenberg groups

In this section we review some of the well-known ties between the Hermite operator and the Heisenberg groups. We present the result by Coulhon, Müller and Zienkiewicz [4] establishing the dimension-free  $L^p$  boundedness of the Heisenberg-Riesz transform. Their estimate involves the  $L^p$  norms of the Hilbert transform  $\mathcal{H}$  along the parabola in the plane. In order to push the approach of [4] to its limit we devote special attention to summarizing a series of deep results by Seeger, Tao and Wright obtained in recent years which together give (almost) sharp  $L^p$  estimates of  $\mathcal{H}$ .

For the purpose of keeping this section as self-contained as possible we need to start with the basic definitions. They can be found in most of the introductory texts on Heisenberg groups, for example [23], [31] or [27].

By the Heisenberg group  $\mathbb{H}^n$  we understand  $\mathbb{R}^{2n+1}$  endowed with the multiplication

$$(u, z) \cdot (u', z') = \left( u + u', z + z' + 2 \sum_{j=1}^n (x'_j y_j - x_j y'_j) \right).$$

Here  $u = (x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n}$ ,  $z \in \mathbb{R}$ , and similarly for  $(u', z')$ .

Let  $\lambda : \mathbb{H}^n \rightarrow \text{Lin}(\mathcal{S}(\mathbb{H}^n))$  be the left regular representation of  $\mathbb{H}^n$ , i.e.  $\lambda : h \mapsto \lambda_h$  where  $\lambda_h f(v) = f(h^{-1}v)$ . An operator  $S \in \text{Lin}(\mathcal{S}(\mathbb{H}^n))$  is said to be *left-invariant* if  $S \circ \lambda_h = \lambda_h \circ S$  for every  $h \in \mathbb{H}^n$ .

The underlying Lie algebra  $\mathfrak{h}^n$  of all left-invariant vector fields on  $\mathbb{H}^n$  is generated by the fields

$$\begin{aligned} X_{2k-1} &= \frac{\partial}{\partial x_k} + 2y_k \frac{\partial}{\partial z} \\ X_{2k} &= \frac{\partial}{\partial y_k} - 2x_k \frac{\partial}{\partial z} \end{aligned}$$

for  $k \in \{1, \dots, n\}$  and

$$Z = \frac{\partial}{\partial z}.$$

The Lie bracket in  $\mathfrak{h}^n$  is defined as  $[U, V]\varphi = U(V\varphi) - V(U\varphi)$ .

In [4] the authors deal with the Riesz transforms on the Heisenberg group, defined as  $X_j \mathcal{L}^{-1/2}$ ,  $j = 1, \dots, 2n$ , where

$$\mathcal{L} = - \sum_{j=1}^{2n} X_j^2$$

is the sub-Laplacian on  $\mathbb{H}^n$ . They prove, with some constant  $C(p)$  independent of  $n$ , that

$$\|\mathbf{R}_{\mathbb{H}^n} f\|_p \leq C(p) \|f\|_p,$$

where

$$\mathbf{R}_{\mathbb{H}^n} f = \left( \sum_j |X_j \mathcal{L}^{-1/2} f|^2 \right)^{1/2}.$$

A feature of their proof is the use of Gaveau-Hulanicki formula for heat kernels on the Heisenberg group which is utilized for representation of the Riesz transforms (see the formula on p. 378 of [4] which, due to a misprint, requires a minor modification; compare with p. 374 there):

$$(X_k \mathcal{L}^{-1/2})f(u) = \int_{\mathbb{H}^n} \Phi_k(v) \tilde{\mathcal{H}}_v f(u) dv,$$

where  $u = (x_1, y_1, \dots, x_n, y_n, s) \in \mathbb{H}^n$ ,

$$\Phi_k = \begin{cases} x_j F - i y_j G & ; \quad k = 2j - 1 \\ y_j F + i x_j G & ; \quad k = 2j \end{cases}$$

and, for  $v = (w, z) \in \mathbb{H}^n$ ,

$$\begin{aligned} F(v) &= \int_{-\infty}^{\infty} \mathbf{p}(v, \lambda) \cosh \lambda d\lambda \\ G(v) &= \int_{-\infty}^{\infty} \mathbf{p}(v, \lambda) \sinh \lambda d\lambda \end{aligned}$$

with

$$\mathbf{p}(v, \lambda) = -\frac{1}{4(2\pi)^{n+1}} e^{-\frac{\lambda}{2}(|w|^2 \coth \lambda - i z)} \left( \frac{\lambda}{\sinh \lambda} \right)^{n+1}.$$

Furthermore,  $\tilde{\mathcal{H}}_v$  is the Hilbert transform along the parabola  $t \mapsto \delta_t(v)$  in  $\mathbb{H}^n$ :

$$(\tilde{\mathcal{H}}_v f)(u) = \text{p.v.} \int_{-\infty}^{\infty} f(u \cdot \delta_t(v)^{-1}) \frac{dt}{t}.$$

Here  $\delta_t$  denotes the Heisenberg dilations  $\delta_t(w, z) = (tw, t^2z)$ . The authors observe that the norms of  $\tilde{\mathcal{H}}_v$  can be reduced to estimating the Hilbert transform along the standard parabola in  $\mathbb{R}^2$ , i.e.

$$(\mathcal{H}f)(x, y) = \text{p.v.} \int_{-\infty}^{\infty} f(x - t, y - t^2) \frac{dt}{t}, \quad (x, y) \in \mathbb{R}^2.$$

By proceeding as in Section III of [4] one arrives at the estimate

$$\|\mathbf{R}_{\mathbb{H}^n} f\|_p \leq C_{q,n} \|\mathcal{H}\|_p \|f\|_p \quad (74)$$

with

$$C_{q,n} = [\sigma(S^n)]^{1/p} \left\| \int_0^\infty X_1 p_1(\delta_r(\omega)) r^{2n+1} dr \right\|_{L^q(d\sigma(\omega))}.$$

Here  $S^n$  stands for the unit sphere in the Korányi norm, given by  $\| (u, z) \|^4 = |u|^4 + z^2$ , while  $\sigma$  is the induced surface measure on  $S^n$ . The Korányi norm is homogeneous with respect to the Heisenberg dilations, meaning that  $\|\delta_r \omega\| = r \|\omega\|$  for any  $\omega \in \mathbb{H}^n$ . Furthermore,  $p_1$  is the heat kernel on  $\mathbb{H}^n$  calculated at the level 1; it is explicitly given by the formula due to Gaveau [9] and Hulanicki [12].

Strictly speaking  $C_{q,n}$  depends on  $n$ , but the authors devote their Section IV to showing that it has a majorant which does not. Actually, a careful examination of their proof reveals that it can even be estimated from above by an *absolute* constant, i.e. one independent of both  $n$  and  $q$ . In order to prove that  $C_{q,n} \leq C$  for some  $C > 0$  and all  $q > 1$ ,  $n \in \mathbb{N}$ , let us attempt rewriting the bottom line on p. 375 from [4] with the use of [4, Lemma 4] but with absolute  $C$  instead of  $C_q$ :

$$\begin{aligned} 2 \frac{\pi^{n-1/2}}{\Gamma(n-1/2)} n^{-1-q/2} &\leq C^q \pi^{nq} 2^{-nq} \frac{\Gamma(n)^q}{\Gamma(n+3/2)^q \Gamma(n/2)^{2q}} \\ &\quad \cdot (4\pi^{n+1/2})^{1-q} \frac{\Gamma(n/2)^{1-q}}{\Gamma(n)^{1-q} \Gamma((n+1)/2)^{1-q}}. \end{aligned}$$

By means of the duplication formula for the  $\Gamma$  function [10, A-5] proving the above inequality is the same as proving

$$\frac{2^{-nq+4q+n-2} \pi^{3(q-1)/2}}{n^{1+q/2}} \cdot \frac{\Gamma(n+3/2)^q \Gamma(n/2)^q}{\Gamma(n-1/2) \Gamma(n/2+1/2)^{3q-2}} \leq C^q.$$

The identity  $\Gamma(x+1) = x \Gamma(x)$  translates this into

$$\frac{2^{-nq+4q+n-2} \pi^{3(q-1)/2}}{n^{1+q/2}} \cdot \frac{(n+1/2)^q (n-1/2) \Gamma(n+1/2)^{q-1} \Gamma(n/2)^q}{\Gamma(n/2+1/2)^{3q-2}} \leq C^q.$$

Due to the Stirling formula this is equivalent to proving

$$\frac{2^{4q-3/2} e^{1/2} \pi^{q-1} (n+1/2)^q (n-1/2)^{n(q-1)+1} (n-2)^{(n-1)q/2}}{n^{1+q/2} (n-1)^{(3q-2)n/2}} \leq C^q,$$

that is,

$$2^{4q-3/2} e^{1/2} \pi^{q-1} \cdot \frac{n-1/2}{n} \cdot \left[ \frac{(n+1/2)^2}{n(n-2)} \right]^{q/2} \cdot \left( \frac{n-1/2}{n-1} \right)^{n(q-1)} \cdot \left( \frac{n-2}{n-1} \right)^{nq/2} \leq C^q.$$

From this type of expression it is clear that indeed there is an *absolute* constant  $C > 0$  such that for all  $n \in \mathbb{N}$  and all  $q$ , the left-hand side is majorized by  $C^q$ .

All said implies the following improvement of (74), i.e. the explicite estimate in the major result of [4] with an absolute constant  $C$ :

$$\|\mathbf{R}_{\mathbb{H}^n} f\|_p \leq C \|\mathcal{H}\|_p \|f\|_p, \quad (75)$$

which is to say that the  $L^p$  norm of the vector Heisenberg-Riesz transform is controlled by the  $L^p$  norm of  $\mathcal{H}$  alone.

### Orlicz spaces and the estimate of $\mathcal{H}$

The  $L^p$  boundedness of  $\mathcal{H}$  has been studied for many years. In 1966 it was proven by Fabes [8] that  $\mathcal{H}$  is bounded on  $L^2$ . Later on Stein and Wainger [24] extended this result to  $L^p$  for  $1 < p < \infty$ . As for  $p = 1$ , the problem of determining optimal boundedness of  $\mathcal{H}$  has been a major challenge in the area. There is, namely, a long-standing conjecture that  $\mathcal{H}$  is of weak type  $(1, 1)$ . This question is still open. A close result in this direction, due to Seeger, Tao and Wright [21], says that  $\mathcal{H}$  maps from  $L \log \log L$  to weak  $L^1$  space. Therefore, for every  $\varepsilon > 0$  it maps from  $L \log^\varepsilon L$  to weak  $L^1$ . By invoking an interpolation argument as done by Tao and Wright [26] we see that it maps  $L \log^{1+\varepsilon} L$  to (strong)  $L^1$ . This chain of implications is completed by a theorem of Tao [25] which implies that  $\mathcal{H}$  is bounded on  $L^p$  with the estimate  $\|\mathcal{H}\|_p \leq C_\varepsilon p^{1+\varepsilon}$ , where  $C_\varepsilon > 0$ .

It is worthwhile putting down the amalgam of the above paragraph and (75):

*For every  $\varepsilon > 0$  there is  $C_\varepsilon > 0$  such that*

$$\|\mathbf{R}_{\mathbb{H}^n} f\|_p \leq C_\varepsilon p^{1+\varepsilon} \|f\|_p. \quad (76)$$

Of the vast literature existing on general types of Radon transforms and their estimates we single out the work by Christ, Nagel, Stein and Wainger [2].

## Schrödinger representations

It is left to explain why (76) implies the same estimate for the Hermite-Riesz transforms  $\mathbf{R}$ .

This is done by means of the *Schrödinger representations* of  $\mathbb{H}^n$ . For a fixed  $\lambda \in \mathbb{R} \setminus \{0\}$  define the operator  $\pi_\lambda : \mathbb{H}^n \rightarrow \mathcal{U}(L^2(\mathbb{R}^n))$  by

$$[\pi_\lambda(x, y, z)f](v) = e^{i\lambda(x \cdot v + \frac{1}{2}x \cdot y + \frac{1}{4}z)} f(v + y).$$

Here  $x, y, v \in \mathbb{R}^n$ ,  $z \in \mathbb{R}$  and the dot denotes the usual Euclidean scalar product in  $\mathbb{R}^n$ . This gives rise to a derived representation, let us call it  $d\pi_\lambda$ , of the Lie algebra  $\mathfrak{h}^n$ . It is defined as follows.

Let  $\gamma : [0, 1] \rightarrow \mathbb{H}^n$  be a  $C^1$  curve such that  $\gamma(0) = 0$  and assume  $\Xi \in \mathfrak{h}^n$  is given by

$$\Xi f(a) = \left. \frac{d}{ds} f(a \cdot \gamma(s)) \right|_{s=0}$$

where  $a \in \mathbb{H}^n$  and  $f : \mathbb{H}^n \rightarrow \mathbb{C}$ . By choosing the coordinate curves

$$\gamma_j(s) = (0, \dots, 0, s, 0, \dots, 0)$$

with  $s$  on the  $j$ -th spot, we obtain vector fields  $X_j$  defined on page 35.

Now we can define  $d\pi_\lambda(\Xi)$  as a linear operator on  $L^2(\mathbb{R}^n)$  determined by the rule

$$d\pi_\lambda(\Xi)\varphi = \left. \frac{d}{ds} \pi_\lambda(\gamma(s))\varphi \right|_{s=0}.$$

It is easy to see that

$$\begin{aligned} d\pi_\lambda(X_{2k-1}) &= \xi_k \\ d\pi_\lambda(X_{2k}) &= \eta_k \\ d\pi_\lambda(Z) &= \zeta, \end{aligned}$$

where, for  $v \in \mathbb{R}^n$ ,  $\xi_k f(v) = i\lambda v_k f(v)$ ,  $\eta_k f(v) = \frac{\partial f}{\partial v_k}(v)$  and  $\zeta f(v) = \frac{i}{4}\lambda f(v)$ .

It is possible to extend  $d\pi_\lambda$  to the *universal enveloping algebra* of  $\mathfrak{h}^n$  (for definitions see [27], [14] or [13]), of which we can think as the unital associative algebra consisting of all left-invariant differential operators on  $\mathcal{H}^n$  with the binary operation being just the composition of operators. Consequently we get

$$d\pi_\lambda(\mathcal{L}) = -\Delta + \lambda^2 |v|^2,$$

i.e.  $d\pi_\lambda(\mathcal{L})$  is the (scaled) Hermite operator.

By applying the method of transference [3, Theorem 2.4] to the representation  $\pi_\lambda$  and a single operator  $X_{2k-1}\mathcal{L}^{-1/2}$ , one can verify that we

get  $d\pi_\lambda(X_{2k-1}\mathcal{L}^{-1/2})$ , which is the same as  $\xi_k L^{-1/2}$ . Similarly,  $X_{2k}\mathcal{L}^{-1/2}$  is transferred to  $\eta_k L^{-1/2}$ . Hence if we transfer the operator  $(X_1\mathcal{L}^{-1/2}, \dots, X_{2n}\mathcal{L}^{-1/2})$  the result is  $(\xi_1 L^{-1/2}, \eta_1 L^{-1/2}, \dots, \xi_n L^{-1/2}, \eta_n L^{-1/2})$ . This is admittedly not the same as  $2(\mathcal{A}_1 L^{-1/2}, \mathcal{A}_1^* L^{-1/2}, \dots, \mathcal{A}_n L^{-1/2}, \mathcal{A}_n^* L^{-1/2})$ , but by (4) the euclidean norms of these two vectors coincide when  $\lambda = 1$ . Since  $\pi_\lambda$  maps elements of  $\mathbb{H}^n$  into contractions, we obtain, again by Theorem 2.4 from [3], that  $\|\mathbf{R}\|_{B(L^p(\mathbb{R}^n))} \leq \|\mathbf{R}_{\mathbb{H}^n}\|_{B(L^p(\mathbb{H}^n))}$ . In words, the  $L^p$  norm of the vector Hermite-Riesz transform is majorized by the  $L^p$  norm of the vector Heisenberg Riesz transform. From (76) we finally get the following:

*For every  $\varepsilon > 0$  there is  $C_\varepsilon > 0$  such that*

$$\|\mathbf{R}f\|_p \leq C_\varepsilon p^{1+\varepsilon} \|f\|_p. \quad (77)$$

**Discussion:**  $1 + \varepsilon$  or 1

We saw three results where the operators involved were estimated in  $L^p$  by  $O((p-1)^{-1-\varepsilon})$ ,  $p \rightarrow 1$ , or  $O(p^{1+\varepsilon})$ ,  $p \rightarrow \infty$ . These are: 1) estimates of the parabola Riesz transform  $\mathcal{H}$ , 2) the estimate (76) of  $\|\mathbf{R}_{\mathbb{H}^n}f\|_p$ , and 3) the last estimate (77) of  $\|\mathbf{R}f\|_p$ .

The logic is that the estimate of  $\mathcal{H}$  implies that of  $\|\mathbf{R}_{\mathbb{H}^n}f\|_p$ , and this, in its turn, implies the estimate of  $\|\mathbf{R}f\|_p$ . On the other hand, we presented here Corollary 1, where the estimate of  $\|\mathbf{R}f\|_p$  is obtained without  $\varepsilon$ , i.e. it is linear. Two natural questions arise:

Q. 1: Is it possible to obtain a linear estimate of  $\|\mathbf{R}_{\mathbb{H}^n}f\|_p$ ?

Q. 2: Is it possible to obtain a linear estimate of  $\|\mathcal{H}\|_p$ ?

We strongly believe that the answer to Q.1 is “yes” and that the answer to Q.2 is “no”.

Let us comment on that. In what concerns Q.1 we believe that the Bellman function approach used in the present paper is capable to treat a very wide range of Riesz transforms – always giving estimates which are linear in  $p$  and dimension-free.

As for Q.2, it has been proved in [26] that a certain class of operators map  $L \log \log L$  into  $L^{1,\infty}$ . Parabola Riesz transform  $\mathcal{H}$  belongs to this class, and now there is a strong feeling that this is sharp. In particular,  $L \log^\varepsilon L$  is mapped to  $L^{1,\infty}$ , and, hence,  $L \log^{1+\varepsilon} L$  to  $L^1$ , see [1], [26]. But one cannot get rid of  $\varepsilon$ , because  $L \log L$  is probably not mapped to  $L^1$ ! However, Yano’s extrapolation theorem [32] implies that the linear estimate of  $\mathcal{H}$  would give  $L \log L$  to  $L^1$  action. So we come to a “contradiction”.

Coming back to Q.1, it seems natural to try to prove an analogue of our bilinear embedding (Theorem 1) in the context of  $\mathbb{H}^n$ . Then one can



hope to have the analog of Corollary 1.

#### 4.4 Sharpness of the linear estimate

We believe that the linear  $p - 1$  estimate of Theorem 1 and its Corollary 1 cannot be improved. There are several examples when similar singular operators got the linear estimate from below. Each time it is a separate and non-trivial task. We believe this should be feasible and will be a subject of future efforts.

## References

- [1] BERGH, J., LÖFSTROM, J.: *Interpolation spaces: An introduction*, Springer Verlag, 1976.
- [2] CHRIST, M., NAGEL, A., STEIN, E. M., WAINGER, S.: *Singular and maximal Radon transforms: analysis and geometry*, Ann. of Math. (2) **150** (1999), no. 2, 489–577.
- [3] COIFMAN, R., WEISS, G.: *Transference methods in analysis*, Regional conference series in mathematics **31**, American Mathematical Society, Providence, RI, (1977).
- [4] COULHON, T., MÜLLER, D., ZIENKIEWICZ, J.: *About Riesz transforms on Heisenberg groups*, Math. Ann. **305** (1996), 369–379.
- [5] DRAGIČEVIĆ, O., VOLBERG A.: *Bellman functions and dimensionless estimates of Littlewood-Paley type*, J. Oper. Theory **56** (2006), no. 1, 167–198.
- [6] DRAGIČEVIĆ, O., VOLBERG, A.: *Bilinear embedding theorem for elliptic differential operators in divergence form with real coefficients*, preprint (2006).
- [7] DUONG, X. T., OUHABAZ, E. M., SIKORA, A.: *Plancherel-type estimates and sharp spectral multipliers*, J. Funct. Anal. **196** (2002), 443–485.
- [8] FABES, E. B.: *Singular integrals and partial differential equations of parabolic type*, Studia Math. **28** (1966), 81–131.

- [9] GAVEAU, B.: *Principe de moindre action, propagation de la chaleur et estimates sous-elliptiques sur certains groupes nilpotents*, Acta Math. **107** (1977), 95–153.
- [10] GRAFAKOS, L.: *Classical and Modern Fourier Analysis*, Pearson/Prentice Hall (2004).
- [11] HARBOURE, E., DE ROSA, L., SEGOVIA, C., TORREA, J. L.:  *$L^p$ -dimension free boundedness for Riesz transforms associated to Hermite functions*, Math. Ann. **328** (2004), 653–682.
- [12] HULANICKI, A.: *The distribution of energy in the Brownian motion in the Gaussian field and analytic-hypoellipticity of certain subelliptic operators on the Heisenberg group*, Studia Math. **56** (1976), 165–173.
- [13] HUMPHREYS, J. E.: *Introduction to Lie algebras and representation theory*, 2<sup>nd</sup> printing, revised, Graduate Texts in Mathematics **9**, Springer-Verlag, New York-Berlin, 1978.
- [14] KNAPP, A. W.: *Lie groups beyond an introduction*, 2<sup>nd</sup> edition, Progress in Mathematics **140**, Birkhäuser Boston, Inc., Boston, MA, 2002.
- [15] LUST-PIQUARD, F.: *Dimension free estimates for Riesz transforms associated to the harmonic oscillator on  $\mathbb{R}^n$* , Potential Anal. **24** (2006), no. 1, 47–62.
- [16] MAUCERI, G.: *The Weyl Transform and Bounded Operators on  $L^p(\mathbb{R}^n)$* , J. Funct. Anal. **39** (1980), 408–429.
- [17] MÜLLER, D., STEIN, E. M.:  *$L^p$ -estimates for the wave equation on the Heisenberg group*, Rev. Mat. Iberoamericana **15** (1999), 297–334.
- [18] NARAYANAN, E. K.: *Multipliers for the twisted Laplacian*, Colloq. Math. **97** (2003), no. 2, 189–205.
- [19] NAZAROV, F., TREIL, S.: *The Hunt for a Bellman function: applications to estimates of singular integral operators and to other classical problems in harmonic analysis*, St. Petersburg Math. J. **8** (1997), no. 5, 721–824.
- [20] NAZAROV, F., VOLBERG, A.: *Heat extension of the Beurling operator and estimates for its norm*, St. Petersburg Math. J. **15** (2004), no. 4, 563–573.

- [21] SEEGER, A., TAO, T., WRIGHT, J.: *Singular maximal functions and Radon transforms near  $L^1$* , Amer. J. Math. **126** (2004), no. 3, 607–647.
- [22] SIMON, B.: *Kato’s inequality and the comparison of semigroups*, J. Funct. Anal. **32** (1979), no. 1, 97–101.
- [23] STEIN, S.: *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, New Jersey (1993).
- [24] STEIN, E. M., WAINGER, S.: *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 1239–1295.
- [25] TAO, T.: *A converse extrapolation theorem for translation-invariant operators*, J. Funct. Anal. **180** (2001), 1–10.
- [26] TAO, T., WRIGHT, J.: *Endpoint multiplier theorems of Marcinkiewicz type*, Revista Mat. Iber. **17** (2001), no. 3, 521–558.
- [27] TAYLOR, M. E.: *Noncommutative harmonic analysis*, Mathematical Surveys and Monographs **22**, American Mathematical Society, Providence, RI (1986).
- [28] THANGAVELU, S.: *Littlewood-Paley-Stein Theory on  $\mathbb{C}^n$  and Weyl Multipliers*, Rev. Mat. Iberoamericana **6** (1990), no. 1-2, 75–90.
- [29] THANGAVELU, S.: *Riesz transforms and the wave equation for the Hermite operator*, Comm. Partial Differential Equations **15** (1990), no. 8, 1199–1215.
- [30] THANGAVELU, S.: *Lectures on Hermite and Laguerre expansions*, Mathematical notes **42**, Princeton University Press (1993).
- [31] THANGAVELU, S.: *Harmonic Analysis on the Heisenberg Group*, Progress in Mathematics **159**, Birkhäuser (1998).
- [32] YANO, S.: *Notes on Fourier analysis. XXIX. An extrapolation theorem*, J. Math. Soc. Japan **3** (1951), 296–305.

OLIVER DRAGIČEVIĆ, Faculty of Mathematics and Physics, University of Ljubljana, and Institute of Mathematics, Physics and Mechanics, Jadranska 19, SI-1000 Ljubljana, Slovenia

`oliver.dragicevic@fmf.uni-lj.si`

ALEXANDER VOLBERG, Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA, and School of Mathematics, University of Edinburgh, Edinburgh EH9 3JZ, UK  
`volberg@math.msu.edu`, `A.Volberg@ed.ac.uk`